TRANSIENTS IN LINEAR SYSTEMS

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Laplace Transformation

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VOLUME I

Lumped-Constant

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PREFACE

This book has evolved from a set of lecture notes used in a one-semester course offered to beginning graduate students in electrical and mechanical engineering at the Massachusetts Institute of Technology. Until 1929 the course was taught by Dr. Vannevar Bush, using at first notes and later his book *Operational Circuit Analysis*. Since 1930 the course has been given by the first-named author of this book. The second-named author wrote his dissertation on the Laplace-Stieltjes transformation and has collaborated on the work in the field of this subject since 1935. He has also introduced at Tufts College a simplified form of the course for junior and senior students.

Primarily the book is intended for first-year graduate work in electrical and mechanical engineering. Secondarily it can be used for graduate work in mathematical physics and applied mathematics. A large part of the text material could be used in undergraduate senior courses for advanced or honor students. Chapter 2, on the mathematical expression of problems concerning lumped-parameter electric and mechanical systems, is in a sense a complete unit and could be used as an adjunct to any engineering course on ordinary differential equations regardless of the methods used in their solution.

It is expected that this book also will serve as a reference for practicing electrical and mechanical engineers and industrial physicists.

As the title suggests, the treatment is limited to linear physical systems. Although the graduate course mentioned above includes a study of systems having distributed parameters, it seemed better to restrict this volume to the study of systems having lumped parameters. Formulated mathematically, systems of the latter type lead to ordinary integrodifferential or difference equations. Distributed-parameter systems which bring in partial integrodifferential equations will be treated in Volume 2. This division of subject matter corresponds to the usual separation of the treatment of differential equations into the ordinary and partial types. Volume 1 is complete in itself despite a number of references to Volume 2. Since in writing Volume 1 we have the extended applications of the method in mind, the techniques developed are not necessarily the ones that would be chosen if the problems solved here were the only ones of interest.

Chapter 1 includes certain basic preliminaries on the nature of transients and the reasons for restricting the study to linear systems. It concludes with a preview of the Laplace-transformation method of solving functional equations.

The mathematical formulation of one-dimensional problems concerning electric and mechanical systems is covered in Chapter 2.

The introduction of the Laplace transformation in Chapter 3 is accomplished through successive generalizations of Fourier series and integrals.

In Chapter 4 elementary functions are transformed and fundamental theorems on the Laplace transformation are set down.

Equations of the types formulated in Chapter 2 are next transformed and solved algebraically in Chapter 5.

In order to find the inverse transforms of rational algebraic functions of the type which arise in Chapter 5, i.e., to fit them into the table of simple transforms, it is necessary to resolve them into partial fractions, and this is covered in Chapter 6.

Finally, in Chapter 7 the formulation and complete solution of typical problems is carried through.

Additional theorems and variations with applications are collected in Chapter 8.

The last chapter, 9, is on integrodifferential difference equations.

Tables of operation and function transforms convenient for use in practical applications of the Laplace-transformation method appear in Appendix A. Additional notes on the relation between the Fourier and Laplace transformations are given in Appendix B. Appendix C contains historical notes on the mathematical theory.

The book concludes with a bibliography containing references to the mathematical theory and to the engineering practice.

Naturally the methods for solving the problems considered have been improved through repeated study by many of those working in this field. As would be expected, the class of readily solvable problems has grown continually. The subject has experienced a considerable change in point of view from the operational methods to that of functional transformations. Many of the explicit and implicit restrictions current in the engineering literature of a few years ago are now obsolete. In the present text we hope, with the aid of recent forms of mathematical theorems, to make available for engineers methods and results which will more closely approach invariance with the passing of time.

An attempt has been made to give the simplest and most practical presentation of advanced mathematical methods for handling not only ordinary integrodifferential equations but also difference and partial

differential equations. This has been accomplished by basing the treatment on a table of transform pairs which is derived and used in much the same way that an ordinary table of integrals is used. By this means the use of the inverse Laplace-transformation integral and its evaluation by integration in the complex plane are postponed until this integral is definitely needed — in fact its use is not required in this volume. The treatment enables engineers to solve many problems by an effective and reliable method and yet avoid the quagmire of complex integration.

In contrast to the conventional operational treatment which is usually formal, i.e., a treatment in which the range of validity is not carefully given, the methods used are based on well-known classical mathematics. The ranges of validity are either given or can be found by classical methods, and the treatment can be made as rigorous as desired.

It has not been our object to use the Laplace-transform theory to justify the earlier operational methods. Rather than this we have aimed to show that transform theory itself with an abbreviated notation provides equally short methods of expression and solution of problems while using only recognized and precise mathematical techniques. We have sought to retain the vigor and simplicity of these earlier operational methods without sacrificing the rigor and effectiveness of the functional-transformation method.

We have tried to make the approach straightforward and simple by avoiding the introduction of mathematical techniques until we are ready to use them. This scheme has enabled us to postpone the more difficult parts of the mathematics. Also for simplicity we have placed all the discussion on a given step in the procedure in a single chapter. An advantage of this arrangement is that ideas can be introduced a few at a time. Practice can then be given on each set of ideas to insure their retention before a new set is introduced. For those who find the ideas introduced too slowly, or who wish to complete quickly the solution of a particular problem, we recommend a rapid perusal of the first seven chapters followed by a more careful study of the details.

A supplementary object has been to indicate where important material on the history of the mathematical theory can be found and to help straighten out some of the historical inaccuracies present in the existing literature.

Mention may be made of certain features of this book. In common with the Campbell and Foster table of Fourier transforms the treatment makes maximum use of mathematical expressions in closed form. Series developments including asymptotic expansions are usually avoided or are reserved for Volume 2.

Emphasis is placed on systematic methods of setting up physical problems in mathematical form.

Through Chapters 1 and 3 and Appendix B appear certain comparisons of the Fourier method with the Laplace method.

Novel features will be found in Chapter 9, which serves as a transition chapter between those chapters treating one-dimensional problems and the chapters of Volume 2 on multi-dimensional problems. In particular, wide use is made of continuously defined jump functions to interpolate functions defined only at discrete points.

A final feature is the large number of illustrative examples presented in detail. At the end of every chapter appear practice problems, many of which have a definite engineering background and were made up from a wide variety of fields. Certain of these problems are more than practice exercises. They suggest extensions beyond the treatment given in the text and form an integral part of the development of the subject.

To Dr. Vannevar Bush we express our appreciation for the inspiration of a lasting interest in this field. We also wish to thank the many students who have contributed to our understanding of the subject through their questions and suggestions. Finally, thanks are due to Professors D. C. Jackson, E. L. Moreland, and H. L. Hazen for their encouragement in the writing of this book.

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Chapter I

INTRODUCTION

In beginning the mathematical study of transients in linear systems it is necessary to have in mind certain basic ideas regarding the nature of transients, and to understand clearly the reasons for restricting the study to linear systems. In this introductory chapter these basic preliminaries are discussed, and reasons are given for the choice of the Laplace-transformation method for the mathematical treatment of linear physical systems in the transient state.

A. THE NATURE OF TRANSIENTS

Mention of several examples of familiar transients will serve as an intuitive introduction to the subject.

After a radio receiver having heater-type vacuum tubes is turned on, approximately half a minute must elapse before it begins to function properly. This interval during which the tube cathodes are changing from "cold" to "hot" is called a transient interval. The thermal and electrical characteristics of the receiver during this time are said to be in a transient state. The transient interval is thus the interval of transition in which the tube cathodes change from the steady "cold" state to the steady "hot" state.

Another example of familiar transient behavior is provided by an electric motor of a refrigerator when starting. Mechanically the rotor makes the transition from the steady state of rest to the steady state of rotation. This change is accompanied by a transition from rest to a steady state of vibration of the motor frame and its supports. Electrically, also, the motor is in the transient state during this interval — its generated electromotive force changing from the constant zero value to a steady operating value.

1. Definition of Steady and Transient States

In the preceding examples it was easy to distinguish that part of the behavior which should be called the steady state and that which should be called the transient state. Unfortunately, this is not the case for all examples. The very changes by which certain important aspects of physical problems are simplified to make these problems solvable often

bring in features which obscure the distinction between the steady and transient states for the simplified versions. In some of the problems, after simplification there is no transient state, i.e., no beginning and no end; or, as the idea is sometimes phrased, the steady-state interval is infinite. Passing to the other extreme, there may be no steady state. For this extreme the statement that the transient state is the transition state from one steady state to another loses significance.

To save the essence of the intuitive ideas of steady and transient state, and yet be sufficiently general, the following definitions are formulated.

A dynamical system is said to be in the *steady state* when the variables describing its behavior are either invariant with time, or are (sections of) periodic functions of time.

A dynamical system is said to be in the *transient* (or unsteady) *state* when it is not in the steady state.

From a physical point of view it may be said that a transient state exists in a physical system while the energy conditions of one steady state are being changed to those of a second steady state.

2. Undesirable Transients

Certain transient phenomena accompanying the redistribution of energy in a physical system are often wholly undesirable yet inescapable accompaniments of such a change.

Among the economically important but unwanted electrical transients are the natural transients, such as lightning and static, and the man-made transients arising from switching. The principal undesired mechanical transients can likewise be divided into the natural transients, such as storms in the air and on the sea and tremors of the earth, and the man-made transients, such as the vibrations produced by the acceleration of machines.

To these electrical and mechanical transients may be added thermal and acoustic transients. The natural thermal transients occurring daily and seasonably greatly affect our lives, and the man-made thermal transients in electrical and mechanical apparatus often impose serious limitations on the use of this apparatus. Unwanted acoustic transients are the object of much concern in the design of sound systems for auditoriums and broadcasting studios.

3. Desirable Transients

In contrast to these uncontrolled and unwanted transients there are desirable and controlled transients which are put to many important

uses. For example, the transient state is the important one for the sound-controlled electric currents in telephone and radio systems, and for key-controlled currents in telegraph and cable systems. Furthermore, the transient state is the all-important one for the acoustic phenomena arising in such systems, vocal sounds and instrumental music being composed principally of transients. Television systems use desirable transients as modulation and timing controls. The practical importance of variable-speed motors rests largely on their transient behavior under changes of load; nearly all land, water, and air transports use such motors.

The basis for self regulation in all self-controlled systems is transient behavior. The importance of such automatic systems for the control of position, speed, voltage, frequency, sound-volume level, temperature, humidity, liquid level, and especially for the control of industrial machines and processes is increasing rapidly.

In brief, many dynamical systems need to be studied in the transient state — some for the purpose of minimizing unwanted transient effects, others for the purpose of controlling and employing desirable effects. But in whichever aspect transients may occur, desirable or undesirable, their treatment is a problem of engineering importance and their calculation often a matter of difficulty.

B. RESTRICTION TO LINEAR SYSTEMS

If it were possible to handle simply and completely by mathematics *any* type of dynamical system in the transient state, we should naturally choose to study first those systems that seem most important. Since this possibility does not exist we elect to treat the type of system that can be handled most simply and completely by mathematics. This is the linear type, which fortunately is a very important type in engineering and physics [FR 1, WE 2].*

4. DEFINITION OF LINEAR BEHAVIOR

A resistor of the type used in a radio receiver exhibits the following characteristic: Over the range of operation for which the resistor is designed, the current in it is proportional to the voltage drop across it. A spring of the type used in a spring balance exhibits a similar characteristic: Over the range for which the spring is designed, the force acting through it is proportional to the change in its length.

A physical element, behaving as the above resistor or spring, is said to behave *linearly over a specified range* of one of the variables used to

* All such symbols refer to the Bibliography.

describe its conduct if in this range its behavior can be represented by a linear equation, i.e., an equation of the first degree. The use of the word "linear" to denote an equation of the first degree, whether it is algebraic, difference, differential, or integral, is an extension from its use to denote a first-degree algebraic equation in exactly two variables which can be represented geometrically by a straight line. For brevity, an element which is used only in the range where its behavior is linear is called a *linear element*.

Most pieces of apparatus do not behave linearly over an indefinitely large range, the transition from linearity to nonlinearity being gradual in certain cases and abrupt in others. For example, the resistance of the resistor mentioned above will change gradually owing to overheating if the current through it is sufficiently increased, or the insulation of the resistor will break down abruptly if the voltage across it is sufficiently increased. Each of these effects restricts the range of linear behavior.

Similar effects restrict the range of linearity of the spring mentioned above. As the loading of the balance is increased the material in the spring will ultimately pass its elastic limit — in other words, the behavior of the spring will finally become nonlinear.

5. Replacement of the Physical System by a Simplified Mathematical System

To make easier the solution of physical problems, the nonlinear ranges of physical elements may be conveniently ignored by substituting simplified counterparts for these elements. These simplified abstract elements are defined to be *linear over all ranges*.

This replacement may appear to be a shocking departure from the physical facts. It is a departure which carries with it an important responsibility for the user when later the conclusions derived from the theory are employed to explain and predict other physical facts. Such simplified abstract elements appear commonly in the theories of the physical sciences, and after they have been introduced, their relation to the original elements tends to be forgotten. For instance, Ohm's law without restriction of range applies to the simplified elements only. The same is true for Hooke's law.

Even if the physical elements under consideration have no range of linearity, a theory based on linear elements will sometimes give rough information which is of value. Suppose that the approximating relation is a linear algebraic equation in two variables. Then the geometric interpretation is that over a "sufficiently" short range a curve may be replaced by a straight line, or, sometimes over longer ranges, that a straight line may replace a curve by acting as an upper or lower bound to that curve.

6. Additive Properties of Linear Mathematical Systems

The linear functions are the simplest of the algebraic functions. They possess the very important property that the sum of two of them is also a linear function. Similarly, the linear equations are the easiest of the algebraic equations to solve, and this simplicity is not lost even when systems of linear algebraic equations are to be solved. Furthermore, the elegant determinant and matrix methods for treating systems of algebraic equations apply only to linear equations.

Carrying this correspondence further, linear (i.e., first-degree) difference, differential, or integral equations are easier to solve than higher-degree equations of the same type. Also, such linear equations possess very important additive properties. With a system of linear equations it is possible to add the separate solutions found by using each driving function alone to get the composite solution with all driving functions present. This principle of adding solutions found under such conditions is called the *principle of superposition.*

Briefly, then, it is the simplicity and consequent additive properties of linear mathematical systems that make possible their complete solution with relative ease.

In contrast to the ease of handling linear mathematical systems, consider the difficulty of treating mathematically those physical systems that are basically nonlinear, i.e., those systems that cannot be represented by linear mathematical systems without loss of their essential characteristics. The principal elements of such physical systems are nonlinear. Examples are coils with magnetic cores, resistors of lightning arrestors, and vacuum tubes used as oscillators. A complete mathematical treatment of complicated physical systems containing such elements is extremely difficult and in most instances impractical.

Nonlinear systems are harder to treat mathematically because their behavior is more complicated than the behavior of linear systems. The functions and equations describing the behavior of nonlinear physical elements do not possess the characteristic additive properties of linear elements; in other words, the principle of superposition does not apply. Without these additive properties the use of series expansions leads to much greater complication. Systems of equations can no longer be treated by the efficient mass-production methods of matric algebra.

The practical methods available for analysis of nonlinear systems are inexact and complicated. They are graphical and hence rough, or require successive approximations and hence are laborious, or are carried out by machines [Bu 2, Ha 11]. The study of nonlinear systems is frequently best carried out by experimental rather than by logical (mathematical) methods. For this models or analogs are particularly suited.

7. Passive Elements Assumed to be Time Invariant

The elements through which energy enters a physical system are called sources or *active elements.* The remaining elements of the system either store energy or withdraw it from the system often in the form of heat; they are called inactive or *passive elements.* To simplify the present treatment, the passive elements considered will be restricted to those which are *invariant with time.*

To make clear the meaning of this last restriction, consider a physical capacitor. Its capacitance may change slightly over a long period of time as a result of slow changes in the dielectric, an effect known as aging; it may change during short time intervals because of rapid temperature changes of its plates and dielectric; or it may be intentionally varied with time by some external means. Such changes in element values with time will, by definition, be absent from all the mathematical systems treated here. In other words, the systems will have constant parameters.

Time changes in a system produced by switching will not be considered to be time changes in the elements of the system. Instead, switching changes will be treated as changes from one system to another, the elements in each system being invariant with time.

Linear systems with invariant elements are of the type designated by the unwieldy phrase of Volterra "systems of the closed cycle" [Vo 2]. Between changes in the connection of the elements, such systems are invariant with time. They possess no "memory" in that sense in which the behavior of human beings, magnetic materials, etc., depends upon whole intervals of their history. It will be seen that the behavior of these time-invariant systems depends only on the condition of the system in the neighborhood of a *single* preceding value of time.

8. Physical Driving Functions Replaced by Simplified Mathematical Functions

The mathematical equations describing the behavior of linear time-invariant systems are linear integrodifferential equations with constant coefficients. It is necessary to consider not only the types of operations but also the types of functions that appear in these equations and in their solutions.

For the same reasons that physical elements are replaced by simpler abstract elements, physical time functions — such as force, velocity, current, and voltage used to drive the system — are replaced by simpler abstract functions.

The physical functions are single valued and, viewed "in the large,"

continuous. We should like to preserve both single-valuedness and continuity in the simplified functions replacing them, but at a vertical step of a simplified function these properties become incompatible. At such a point the single-valuedness is retained and the continuity dropped because, at present, a jump discontinuity is easier to handle than a vertical multivalued line-segment. Frequently, however, the difficulty is avoided by ignoring the behavior of the simplified function at a step.

The simplified functions that are used to replace the physical driving functions belong to the class of functions having bounded variation [Ho 1, p. 325] in every finite interval. The choice of this class of functions has been made on the basis of the following considerations [GI 5]. In our work with functions coming from physical problems we have encountered no functions which lie outside this class. In particular, we have found no need for functions which are Lebesgue measurable [Ho 1, p. 562] but not of bounded variation in finite intervals. The properties of functions having bounded variation in every finite interval are widely discussed in the mathematical literature. We know of no generally studied, smaller, and more special class that includes all functions of the type which we propose to use.

9. Importance of Linear Time-Invariant Systems

Having settled upon abstract linear time-invariant systems as the simplified models to be studied — for reasons which may be summarized under the term expediency — the question of the prevalence and importance of physical systems that can be properly approximated by such simplified systems arises.

To speak precisely about approximations requires that a measure of the closeness of approximation, i.e., the tolerance or allowable error, be known. This measure depends, of course, on the particular use to be made of the approximation. Furthermore, compared with the approximate linear time-invariant system, other systems that fit the physical system more closely usually require for their use an amount of labor that is less tolerable than the roughness of approximation introduced by the use of the linear system.

One can only guess the answer to the above question on the importance of linear time-invariant systems, but it is our impression that all but a small percentage of the physical systems now susceptible to complete treatment by mathematics can be handled to a first approximation by such abstract systems.

Another aspect of the importance of mathematical systems is the

following: When passage is made from physical to mathematical systems it is discovered that many different physical systems lead to the same abstract system, i.e., to the same mathematical problem. Such physical systems are called *analogs*. The correspondences of analogous physical systems with each other and with the mathematical system, if fully used, result in an enormous saving of work. The solution of one abstract problem carries with it the solution of all its physical representations. The importance and usefulness of linear time-invariant mathematical systems will be illustrated by the examples beginning in Chapter 2.

C. CHOICE OF THE LAPLACE-TRANSFORMATION METHOD

10. Comparison of Four Available Methods

The problems to be solved are formulated mathematically as systems of linear constant-coefficient integrodifferential equations with driving functions having bounded variation in every finite interval and with general initial or boundary conditions. There are four prominent methods for solving the simpler problems of this type. The most widely known of these methods is the one which appears in all elementary texts on ordinary differential equations. It will be called the classical method. The other three are the Cauchy-Heaviside operational method [Appen C], the Fourier-transformation method, and the Laplace-transformation method.

None of these methods has been developed as far as it might be with regard to extent and conditions of applicability, simplicity of use, and clearness of exposition. In the opinion of the authors, the Laplace-transformation method, although least widely known, is at present the best general method and offers the greatest promise for further improvement.

Table 1 shows an estimated rating of these methods, based on the forms that are in current use. The following remarks supplement this table.

In certain respects the four methods are very much alike, and in particular the Cauchy-Heaviside, Fourier, and Laplace methods are but different aspects of a single method.

A study of the Cauchy-Heaviside (C-H) type of operational calculus shows that it is considerably limited in field of application in its present form and is incapable of interpretation or growth without help from less symbolic (formal) branches of mathematics. Cauchy's early work [Appen C] in this type of operational calculus shows that he had some appreciation of its close relation to the Laplace-transformation method. Heaviside ignored this relation, and since his period Giorgi, Bromwich,

TABLE 1. COMPARISON OF FOUR AVAILABLE METHODS

Method	Additional Restrictions on Range of Practical Applicability	Conditions of Validity	Simplicity from Standpoint of Use	Possibility of Further Development
Classical△	(1) Equations in one independent variable. (2) Small number of integration constants.† (3) If equations contain integrals, then driving functions must be differentiable. (Includes a few unimportant functions excluded by other methods.)	Available but not widely known. They can be determined by standard mathematical methods.	Not very simple because, unless the number of constants of integration is small, it is not easy to determine these constants.	Great, but development hardly warranted because of inherent shortcomings now present.
Cauchy-Heaviside	(1) Boundary conditions: initial values all zero or artifices must be used. (2) No interpretation for multiplication of functions of t. (3) Method does not provide information on range of applicability.	Method does not provide them. Fragmentary conditions obtained from analogy with Laplace-transformation method.	Excellent.	Very low, unless it is done by analogy with Laplace-transformation method.
Fourier-transformation	(1) No provision for finite boundary conditions.* (2) Driving and solution functions must have Fourier transforms. (Permits jump discontinuities, but excludes many useful functions.)	Precise conditions are known.	Excellent.	Limited because of inherent restrictions on transformable functions.
Laplace-transformation	(1) Driving and solution functions must have Laplace transforms. (Includes all functions allowed by Cauchy-Heaviside and Fourier-transformation methods. Includes many functions not allowed by classical method.)	Precise conditions are known.	Excellent.	Great.

△ See Sec. 10, Chapter 1. † See [Bu 1, pp. 51–53]. * See [Bo 1, Chapter 5].

Wagner, Carson, von Stachó, Doetsch, van der Pol, Plancherel, Maechler, and others have used some aspect of the Laplace transformation in attempting to make rigorous the C–H operational calculus. Today it is quite clear that the Laplace-transformation method provides a rigorous substantiation and a clear interpretation of the symbolic (formal) C–H operational calculus. With the aid of this point of view, one not only understands why the C–H calculus proceeds as it does, but one also can see extensions along lines which were otherwise imperceptible. A set of dissociated rules of procedure is replaced by a closely connected systematic method.

In this book the symbolic C–H operational calculus has been abandoned in favor of the rigorous supporting transformation method. Furthermore, no attempt will be made to trace the parallelism between the two methods.

Compared with the classical method, the Laplace-transformation method requires more knowledge of mathematics but gives a deeper insight into the relation between the transient and steady-state parts of the solution. The simplification and unification of the process that result from the use of more advanced ideas more than offset the work required to learn these ideas.

In Chapter 3 it is shown how the Fourier-transformation method is included as a special case in the Laplace-transformation method.

This book will present the salient features of the Laplace-transformation ($\mathfrak{L}$-transformation) method, and will indicate the types of problems whose solution is simplified by its use. Mention will also be made of certain types of problems, such as those with two-point boundary conditions, whose solution is not much simplified by use of the method.

11. Features of the $\mathfrak{L}$-Transformation Method

To those interested in solving problems concerning linear time-invariant systems the $\mathfrak{L}$-transformation method offers six prominent features.

a. It provides a straightforward procedure for obtaining analytically the solutions of practical problems on transients for general one-point initial or boundary conditions. The solution starts with the integro-differential equations and arrives directly at the desired result for the particular problem set. This is in contrast to the classical scheme in which a general solution is first obtained and then specialized to fit the particular problem under consideration. Likewise it is in contrast to the usual C–H operational calculus solution which is restricted to initial conditions corresponding to zero initial energy storage in the system and which requires various artifices to handle more general initial or boundary conditions.

b. It provides a convenient way of organizing and summarizing previous experience in convenient tabular form, i.e., in tables of function-transform pairs and of operation-transform pairs.

c. It provides by the same process both the transient and steady-state solutions of problems and brings out clearly their close relation.

d. It provides a single method for solving not only ordinary differential equations but also linear constant-coefficient difference equations, partial differential equations, and certain types of integral equations.

e. It provides the basis for both the analysis of existing systems and the synthesis (design) of proposed systems satisfying prescribed requirements.

f. Finally, it is capable of yielding the transient solution without the introduction of the integrodifferential equations of the system in much the same way that the complex number method, which uses complex algebra and is well known in electrical engineering, is capable of yielding the steady-state solution without the introduction of these equations.

12. Representative Engineering Fields in which the $\mathfrak{L}$-Transformation Method Is Useful

The types of problems in which the $\mathfrak{L}$-transformation method is useful are many and varied. A partial list should include studies of:

a. Waveforms of high-voltage or high-current surge ("impulse") generators.

b. Transients in linear vacuum-tube amplifier circuits.

c. Systems self-controlled by transient feedback (servo-control; automatic control of temperature, voltage, angular position, speed, liquid level, sound-volume level, frequency, etc.).

d. Transient vibrations in mechanical systems (mechanical filters, seismographs).

e. Transient vibrations in electromechanical systems (microphones, loud speakers, piezoelectric crystals).

f. Cathode-ray oscillograph circuits (time-delay, beam-sweep, scanning, recurrent-surge).

g. Transients in power-control relay circuits.

h. Short-circuit transients in a-c machines.

i. Transients in arc-welding generators.

j. Transients in systems subject to cyclic switching.

k. Transient disturbances in the motion of an airplane.

l. Transients in wave filters and artificial lines.

m. Traveling-wave transients on transmission lines and cables caused by lightning, switching, and faults (reflections at junctions and terminals).

n. Circuit recovery voltages after current interruption.

o. Surge protection of transformers and rotating machinery; transformer oscillations.

p. Propagation of signals in communication systems (voice transients, keying transients, synchronizing and picture-element pulses in television circuits).
q. Transient inductive interference in communication lines caused by disturbances in neighboring electric power lines.
r. Transients in sound-wave propagation.
s. Transients in output circuits of loud-speaker amplifiers.
t. Transient transverse or longitudinal vibrations in bars.
u. Transient heat flow in machinery, insulation, cables, metal-to-glass seals, steel ingots.
v. Magnetic field transients in solid iron cores (influence on rapid excitation of large a-c generators, and rapid reversing of d-c rolling-mill motors).

D. BRIEF PREVIEW OF THE $\mathfrak{L}$-TRANSFORMATION METHOD

The following is a brief formal introduction to the $\mathfrak{L}$ transformation and its use in solving integrodifferential equations. The ranges and conditions of validity are covered in the careful development which appears later in the book.

13. THE $\mathfrak{L}$ TRANSFORMATION

The Laplace transformation is a functional transformation; it may also be called a functional operator. It transforms a certain class of functions of a real variable into functions of a complex variable that are "analytic in half-planes or strips." This expression will be explained later. Usually a function and its transform are quite different in form.

The direct $\mathfrak{L}$ transformation is written

$$\int_0^{\infty} f(t)e^{-st}dt = F(s), \tag{1}$$

in which $s \triangleq \sigma + j\omega$, $j \triangleq \sqrt{-1}$ ($\triangleq$ means "equal by definition") and t, σ, and ω are real variables. Equation 1 is abbreviated

$$\mathfrak{L}[f(t)] = F(s). \tag{2}$$

This integral transformation provides a constructive means for determining the transform function $F(s)$ corresponding to the original function $f(t)$.

This functional correspondence can be conveniently exhibited by a table of function-transform pairs as follows:

FUNCTION-TRANSFORM PAIRS

Original	Transform
$f(t)$	$F(s)$

In *constructing* such a table of pairs, the *direct* transformation is used to pass from a function $f(t)$ in the column of "Originals" to its corresponding $F(s)$ or mate in the column of "Transforms." In *using* such a table of pairs, however, one frequently moves in the opposite direction, i.e., from a function $F(s)$ in the column of "Transforms" to its correspondent or mate in the column of "Originals," and in so doing finds the *inverse* Laplace transform of $F(s)$. The inverse Laplace transformation is indicated by

$$\mathfrak{L}^{-1}[F(s)] \; (=) \; f(t), \qquad 0 \leqq t. \tag{3}$$

The sign $(=)$ means "equals almost everywhere" and will be explained later. The restriction to the non-negative range of t is in agreement with the range of integration from 0 to ∞ used in the direct transformation shown in equation 1.

14. Simplification of Functions through Transformation

Many functions of a real variable are carried over into simpler functions of a complex variable by the direct transformation. For instance, functions with step discontinuities transform into functions analytic in a half-plane, and many of the frequently occurring transcendental functions transform into algebraic functions.

As an example of a function with a step discontinuity, consider

$$f_1(t) \triangleq \begin{cases} 0, & t < a, \\ 1, & a < t, \end{cases} \tag{4}$$

in which a is a non-negative real number. For this step function,

$$\mathfrak{L}[f_1(t)] = \int_a^\infty e^{-st} dt = \frac{e^{-st}}{-s} \tag{5}$$

in the domain $0 < \mathcal{R}[s]$, in which $\mathcal{R}$ means "the real part of." It can be seen that no derivative of $f_1(t)$ exists at $t = a$, whereas all derivatives of its transform e^{-as}/s exist in the complex half-plane $0 < \mathcal{R}[s]$.

As an example of a transcendental function, consider $e^{-\alpha t}$, $0 \leqq t$, with α a real number. Here

$$\mathfrak{L}[e^{-\alpha t}] = \int_0^\infty e^{-\alpha t} e^{-st} dt = \int_0^\infty e^{-(s+\alpha)t} dt = \frac{1}{s+\alpha}, \tag{6}$$

in the domain $-\alpha < \mathcal{R}[s]$. Whereas the exponential is transcendental, its transform $(s + \alpha)^{-1}$ is algebraic.

These results are given below as entries in a table of *function*-transform pairs. In addition a third pair is included. It is obtained from the

first by setting $a = 0$, or from the second by setting $\alpha = 0$. It could also be found by using the direct transformation.

FUNCTION-TRANSFORM PAIRS

$f(t)$	$F(s)$
$\begin{cases} 0, & t < a \\ 1, & a < t \end{cases}$	$\frac{1}{s} e^{-as}, \quad 0 \leqq a, \quad 0 < \mathcal{R}[s]$
$e^{-\alpha t}, \quad 0 \leqq t$	$\frac{1}{s+\alpha}, \quad -\alpha < \mathcal{R}[s]$
$1, \quad 0 \leqq t$	$\frac{1}{s}, \quad 0 < \mathcal{R}[s]$

15. Simplification of Operations through Transformation

Besides its property of simplifying certain functions, the $\mathfrak{L}$ transformation has the much more important property of simplifying certain operations. To facilitate the discussion, the following notation is introduced:

$$f'(t) \triangleq \frac{df(t)}{dt};$$

and

$$\int_0^t f(t)dt \triangleq f^{(-1)}(t) - f^{(-1)}(0),$$

or

$$f^{(-1)}(t) = \int_0^t f(t)dt + f^{(-1)}(0)$$

which is sometimes written $\int f(t)dt$.

Using integration by parts, two important relations can be derived which will later be stated as theorems. The first is the transformation of the first derivative:

$$\mathfrak{L}[f'(t)] = sF(s) - f(0). \tag{7}$$

The second is the transformation of the first integral:

$$\mathfrak{L}[f^{(-1)}(t)] = \frac{1}{s}[F(s) + f^{(-1)}(0)]. \tag{8}$$

These two relations show that differentiation and integration (opera-

tions of analysis) transform respectively into multiplication and division (operations of algebra).

A third important relation, namely, the linearity of the direct transformation, follows from the same property of integrals of finite sums of functions. It is

$$\mathfrak{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s), \tag{9}$$

in which a_1 and a_2 may be real or complex numbers.

These relations, equations 7 to 9, can be summarized in a table of *operation*-transform pairs as shown below.

OPERATION-TRANSFORM PAIRS

$f(t)$	$F(s)$
$a_1f_1(t) + a_2f_2(t)$	$a_1F_1(s) + a_2F_2(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(-1)}(t)$	$\frac{1}{s}[F(s) + f^{(-1)}(0)]$

16. SAMPLE SOLUTION OF AN ABSTRACT PROBLEM BY THE $\mathfrak{L}$-TRANSFORMATION METHOD

Both the properties of function-simplification and operation-simplification possessed by the $\mathfrak{L}$ transformation are used in solving an initial-value problem involving an integrodifferential (i-d) equation. To show this, the complete solution of the simple linear constant-coefficient i-d equation,

$$\frac{dy}{dt} + 7y + 12\int y\,dt = e^{-2t}, \quad y \triangleq y(t), \tag{10}$$

with the initial conditions,

$$y(0) \triangleq 2, \quad y^{(-1)}(0) \triangleq 1,$$

will be found.

As a preliminary, assume that the solution $y(t)$ has an $\mathfrak{L}$ transform, and let it be denoted by $Y(s)$. Multiplying each side of equation 10 by e^{-st} and indicating an integration from 0 to ∞,

$$\int_0^\infty \left[\frac{dy}{dt} + 7y + 12\int y\,dt\right] e^{-st}dt = \int_0^\infty e^{-2t}e^{-st}dt. \tag{11}$$

In abbreviated notation this is written

$$\mathfrak{L}\left[\frac{dy}{dt} + 7y + 12\int y\,dt\right] = \mathfrak{L}[e^{-2t}]. \tag{12}$$

From the function-transform table the result of carrying out the indicated operation in the right member of equation 12 is $(s + 2)^{-1}$. From the operation-transform table, using the first pair, the left member of equation 12 becomes

$$\mathfrak{L}[y'] + 7\mathfrak{L}[y] + 12\mathfrak{L}[y^{(-1)}].$$

Then using the second and third pairs, this left member becomes

$$[sY(s) - y(0)] + 7Y(s) + \frac{12}{s}[Y(s) + y^{(-1)}(0)].$$

Substituting the initial values, and equating the left and right members, equation 12 reduces to

$$\left(s + 7 + \frac{12}{s}\right)Y(s) = \frac{1}{s+2} + 2 - \frac{12}{s}. \tag{13}$$

Solving this algebraic equation for $Y(s)$, and then expanding $Y(s)$ in partial fractions,

$$Y(s) = \frac{\dfrac{1}{s+2} + 2 - \dfrac{12}{s}}{s + 7 + \dfrac{12}{s}} = \frac{2s^2 - 7s - 24}{(s+2)(s+3)(s+4)}$$

$$= \frac{K_1}{s+2} + \frac{K_2}{s+3} + \frac{K_3}{s+4}, \tag{14}$$

in which

$$K_1 \triangleq [(s+2)Y(s)]_{s=-2} = \frac{8 + 14 - 24}{(1)(2)} = -1$$

$$K_2 \triangleq [(s+3)Y(s)]_{s=-3} = \frac{18 + 21 - 24}{(-1)(1)} = -15$$

$$K_3 \triangleq [(s+4)Y(s)]_{s=-4} = \frac{32 + 28 - 24}{(-2)(-1)} = 18.$$

The inverse transformation of equation 14 is indicated by

$$\mathfrak{L}^{-1}[Y(s)] = \mathfrak{L}^{-1}\left[-\frac{1}{s+2} - \frac{15}{s+3} + \frac{18}{s+4}\right]. \tag{15}$$

But the left member $\mathfrak{L}^{-1}[Y(s)]$ $(=)$ $y(t)$ in accordance with the earlier assumption. By using the first pair of the operation-transform table from right to left, i.e., inversely, the right member of equation 15 becomes

$$-\mathfrak{L}^{-1}\left[\frac{1}{s+2}\right] - 15\mathfrak{L}^{-1}\left[\frac{1}{s+3}\right] + 18\mathfrak{L}^{-1}\left[\frac{1}{s+4}\right].$$

Applying the second pair in the function-transform table from right to left, the right member becomes $(=)\ -e^{-2t} - 15e^{-3t} + 18e^{-4t},\ 0 \leqq t$. Consequently the final result is

$$y(t)\ (=)\ -e^{-2t} - 15e^{-3t} + 18e^{-4t}, \quad 0 \leqq t. \qquad [16]$$

In this complete solution the first term is the forced solution and the last two terms constitute the free solution. To verify the solution it must be shown that the equation is satisfied and that the initial conditions are fulfilled.

The initial value of $y(t)$ is checked readily; it is

$$y(0) = -1 - 15 + 18 = 2.$$

Unfortunately $y^{(-1)}(0)$ cannot be checked similarly. The reason for this is that $y^{(-1)}(t)$ is equal (page 14) to the normalized inverse derivative of $y(t)$, plus an unspecified constant of integration. Since the value of this constant is determined by the given initial condition against which a check is sought, one cannot say at the end that any agreement constitutes a check on the solution. In other words, integrating $y(t)$ will not provide a check that the solution fits $y^{(-1)}(0)$ because the operation of integration is only unique to within an additive constant. However, a second check on $y(t)$ can be obtained by differentiating it and comparing the value of $y'(0)$ with that computed from the i-d equation using the prescribed initial conditions. Thus here

$$y'(t) = 2e^{-2t} + 45e^{-3t} - 72e^{-4t},$$

yielding for $t = 0$,

$$y'(0) = 2 + 45 - 72 = -25.$$

This agrees with the initial value of the derivative obtained from the i-d equation by solving for the derivative and setting $t = 0$ as follows:

$$y'(0) = \left(e^{-2t} - 7y - 12\int y\,dt\right)_{t=0} = 1 - (7 \times 2) - (12 \times 1) = -25.$$

Substitution of the explicit form of $y(t)$ in the i-d equation reduces that equation to an identity, thus showing that $y(t)$ is the solution sought.

17. Restriction to Physical Problems in One Independent Variable

The treatment in Chapters 1 to 8 will be restricted to physical systems whose behavior can be described by one independent variable, usually time, and any finite number n of dependent variables (functions). Although the methods are theoretically valid for any value of n, the

labor of carrying out a numerical solution increases rapidly with increase in size of n. Hence, except where there is a recurrent structure, attention will be confined to systems in which n is small and numerical results can be obtained without excessive work.

In keeping with the conventional terminology for differential equations, the adjective "ordinary" is extended to describe integrodifferential equations in one independent variable. It follows that only *ordinary* i-d equations and systems of such equations will be used here with the exception of Chapter 9. There problems concerning repeated-structure electric networks will bring in *ordinary* linear difference and i-d difference equations. I-d difference equations are intermediate between ordinary differential equations and partial differential equations, and serve as a logical transition to partial differential equations to be treated in Volume 2.

PROBLEMS

The solution of algebraic equations will be a necessary step in the complete solution of certain numerical problems in the later chapters, and it will be presumed that methods of solving these equations are known. The reader who is not familiar with the methods available should anticipate this step with supplementary reading [Do 16, KU 4, WH 7, Sc 1] and the solution of several of the practice problems listed below. See also the remarks on page 165.

Find the roots of these algebraic equations:

1·1. $s^3 + 18.53s^2 + 575s + 1{,}243 = 0.$

1·2. $s^3 + 8.22s^2 + 157s + 231 = 0.$

1·3. $s^3 + 1.25 \times 10^7 s^2 + 6.38 \times 10^{13} s + 3.16 \times 10^{14} = 0.$

1·4. $s^3 + 1.64 \times 10^3 s^2 + 6.95 \times 10^5 s + 8.46 \times 10^7 = 0.$

1·5. $s^3 + 75s^2 + 2.0 \times 10^3 s + 1.83 \times 10^4 = 0.$

1·6. $s^4 + 1.43 \times 10^4 s^3 + 268.5 \times 10^6 s^2 + 1.58 \times 10^{12} s + 1.86 \times 10^{15} = 0.$

1·7. $s^4 + 1.18 \times 10^5 s^3 + 4.39 \times 10^9 s^2 + 6.05 \times 10^{13} s + 2.98 \times 10^{16} = 0.$

1·8. $s^4 + 3.60 \times 10^6 s^3 + 7.63 \times 10^{12} s^2 + 2.12 \times 10^{17} s + 3.56 \times 10^{21} = 0.$

NOTE: As a preliminary it is usually desirable to simplify the coefficients. This can be done by making the change of variable $s \triangleq 10^n u$ with n chosen so as to eliminate as far as practicable powers of 10 from the coefficients in the new equation in u.

CHAPTER II

THE MATHEMATICAL EXPRESSION OF ONE-DIMENSIONAL PROBLEMS CONCERNING ELECTRIC AND MECHANICAL SYSTEMS

Experience has shown that one of the steps most difficult to make in the solution of problems in transients is the expression mathematically of the relations among variables present in the physical system. It will avail little to master an effective method of solving differential equations if one cannot first formulate the equations. It is essential that one be able to identify the specific physical principles involved in the problem, and on the basis of them be able to make precise statements of the relations among the variables. Mathematically, many of these statements take the form of differential equations.

The purpose of the present chapter is to show convenient choices of variables, and systematic ways of setting up the differential equations and expressing the initial conditions for electric and mechanical systems. A supplementary objective is to point out the basis of analogs. The present chapter will be limited to the mathematical expression of problems; the method of solution will follow in subsequent chapters. This equation formulation is essentially a review, for use later in text and problems, of certain elementary principles of electric network theory and of dynamics.

A distinction will be made between one-dimensional problems and multi-dimensional problems. Dimension here refers to the number of independent variables necessary to specify the problem adequately. If the dependent variables of the problem can be expressed in terms of only one independent variable, the problem is called a *one-dimensional problem.* If two or more independent variables are required, it is a *two-* or *multi-dimensional problem.* This is a distinction between what may be called network problems and field problems. In network problems, the elements of the entire system are considered lumped, and functional relations are expressed by ordinary differential equations. In field problems, the elements of the system, or at least important parts of it, are considered distributed, and certain of the functional relations take the form of partial differential equations. In this chapter only one-

dimensional problems will be considered; the treatment of two-dimensional problems will be reserved for Volume 2. Furthermore, discussion in this chapter will be limited to problems of electric and mechanical systems. They are essentially problems in dynamics; transients are characteristic of their behavior when a redistribution in energy takes place in the system; time is their independent variable.

In this volume only linear systems having constant lumped elements will be considered. This restriction, however, does not preclude the treatment of systems changing by finite steps. For example, if an electric network is altered by a sequence of switching operations, the behavior of the system during each interval between steps can be treated separately, the terminal conditions for one interval becoming the initial conditions for the succeeding interval. As a result of these limitations the equations will always be ordinary differential or integro-differential equations with constant coefficients, and the terminology " differential " or " i-d equations " used hereafter will imply this unless the contrary is specifically stated.

In this text the term " constant " rather than the term " parameter " will be used to emphasize the fact that the systems treated are constant for the duration of the period being considered. Within the scope of the text it is not necessary to provide for " varying parameters."

The mathematical expression of a physical problem involves recognition of the assumptions being made, selection of the appropriate variables, establishment of a system of reference for the variables, and the application of relevant physical principles. These points are brought out in treating a number of examples in the sections that follow. Units will not be indicated for the variables or the equations unless the problem involves numerical values. With the units chosen from any single system [HA 3] the equations are complete as written.

A. LUMPED-CONSTANT ELECTRIC NETWORKS

Electrical problems are in the closest analysis field problems, i.e., they require both time and space variables. But fortunately in many cases of practical interest, it is justifiable to disregard the space variables and to assume that any disturbance takes place instantaneously throughout the system. In other words, the system may be considered to present a network rather than a field problem, and its constants may be considered to be lumped.

For steady-state analysis, it is customary to assume a system to have lumped constants if the important wavelengths of the currents to be considered are large compared to the maximum physical dimension of the system. For example, the dimensions of a network that might be

assembled on a table top would be considered small in comparison with the wavelength of a current of 100 cycles per second, the wavelength of which is approximately 3000 kilometers. But this same network would not be considered small in comparison with the wavelength of a current of 100 megacycles per second, the wavelength of which is approximately 3 meters.

Unfortunately in dealing with transient disturbances, in which the rate of rise of applied voltages and currents is likely to be large, practically the entire range of frequencies must be considered, and the foregoing criterion cannot be extended to apply directly. It serves, however, to single out those cases in which the distributed character of the system makes it essential to include in the approximating network certain lumped elements to represent field effects that otherwise would be neglected.

1. NETWORK ELEMENTS

Electric networks are composed of active and passive elements. The active elements are the energy sources or generators; the passive elements are the resistors, capacitors, and inductors.

The sources are of two kinds: voltage sources and current sources. This is a mathematical distinction, but it is based on the physical fact that certain sources maintain nearly constant the waveform and magnitude of the internal emf while supplying a variable current, and other sources maintain nearly constant the waveform and magnitude of the internal current while supplying a variable terminal potential difference. The small internal series resistance or inductance of the voltage source may be lumped with the external circuit, or neglected, depending on the precision of representation. Similarly, the small internal shunt-conductance or capacitance of the current source may be lumped with the external circuit or neglected. Sources will be represented diagrammatically as shown in Fig. 2·1.

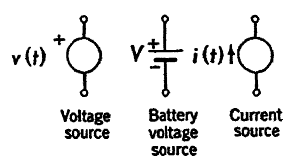


FIG. 2·1. Active elements in electric networks.

In this text v will be used as the symbol to denote both emf and potential difference in order that the symbol e may be retained in its customary mathematical role of Napierian base. The v will be associated with voltage; and it will be convenient to use the terminology "voltage source." The $\pm$ signs indicate the polarity of the source at any instant at which $v(t)$ has a positive value.

The arrow on the current source indicates the direction of flow of

positive charge at any instant at which $i(t)$ has a positive value. In other words, it specifies the direction of positive current in the source.

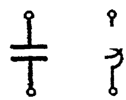
Resistance R is the passive element symbolizing an energy sink; it is an element in which the nonreversible transformation of electric energy into heat takes place. Its value is considered to be invariant with time, and to be uninfluenced by the current which it conducts, or the voltage drop across it. The voltage drop across R in the direction of positive current in R is

$$v_R(t) = Ri_R(t). \qquad [1]$$

If this equation is solved for current,

$$i_G(t) = Gv_G(t), \qquad [2]$$

in which $G \triangleq R^{-1}$ and is called conductance. The diagrammatic representation for both R and G is shown in Fig. 2·2–a.



(b) (c)

FIG. 2·2. Passive elements in electric networks.

Elastance S is the passive element symbolizing an electric-energy reservoir; it is an element in which electric energy is stored. Its value is considered to be invariant with time and to be uninfluenced by the charge on it, or by the voltage drop across it. If the charge on the elastance is represented by $\int i_S(t)dt$, the voltage drop across S in the direction of positive current is

$$v_S(t) = S\int i_S(t)dt \triangleq S\int_0^t i_S(t)dt + v_S(0). \qquad [3]$$

If equation 3 is differentiated once and solved for current,

$$i_C(t) = C\frac{dv_C(t)}{dt}, \qquad [4]$$

in which $C \triangleq S^{-1}$ and is called capacitance. The diagrammatic representation for both S and C is shown in Fig. 2·2–b.

Self-inductance L is the passive element symbolizing a magnetic-energy reservoir; it is an element in which magnetic energy is stored. Its value is considered to be invariant with time and to be uninfluenced by the current in it or the voltage drop across it. The voltage drop across a self-inductance L in the direction of positive current when this current has a positive derivative is

$$v_L(t) = L\frac{di_L(t)}{dt}. \qquad [5]$$

If equation 5 is integrated once and solved for current,

$$i_\Gamma(t) = \Gamma \int v_\Gamma(t)dt \triangleq \Gamma \int_0^t v_\Gamma(t)dt + i_\Gamma(0), \qquad [6]$$

in which $\Gamma \triangleq L^{-1}$ and is called inverse self-inductance. It will be shown later (Sec. 13) that inverse self-inductance is the reciprocal of the self-inductance only when the coupling to other inductances through mutual inductance is zero. Since mutual inductance is not an element, it is not discussed at this point. (See Sec. 7.) The diagrammatic representation for both L and Γ is shown in Fig. 2·2–*c*.

Just as with the sources, pure R, S, and L elements are mathematical fictions since no one of them occurs singly. For example, a coil of conducting wire has resistance, self-inductance, and distributed capacitance. For low-frequency currents its capacitance may be neglected. If it is designed so as to maximize its resistance and minimize its inductance, it serves as a resistor; if the reverse is done, it serves as an inductor. It may be designed to be predominantly one or the other, but it is never solely R or solely L. Furthermore at high frequencies, because of distributed capacitance, it may exhibit more the characteristic of an S than of an L. A capacitor affords a similar example. Here it is the capacitance that is maximized, the conductance in the dielectric and the resistance and inductance of the terminal connections being minimized.

2. Terms Descriptive of Network Geometry; Reference System for Variables

The two terminals of any element are called nodes. A single two-terminal element or a series connection of such elements forms a branch; furthermore, the elements forming a branch may be different in kind e.g., sources and R's. The connection of two elements results in the superposition or coincidence of two nodes forming a single node.

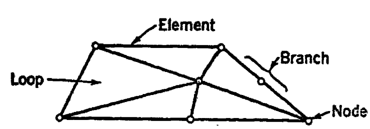


FIG. 2·3. Network branches form a geometric pattern.

The various sources and passive elements when connected conductively, capacitively, or inductively form a network. Viewed purely geometrically the skeleton of a network consists of lines representing the branches which join or intersect at nodes to form a geometric pattern (Fig. 2·3). If all the lines can be mapped on a plane without a crossing, the network is planar.

Any closed path by way of one or more branches in series constitutes a loop. In particular, a single branch when closed on itself forms a loop with at least one node.

A single branch or loop, or a group of connected loops, having pure inductive coupling but no conductive or capacitive connection with other parts of the network is called a separate part. Six examples of geometric patterns are shown in Fig. 2·4. In *a*, *b*, *c*, *d*, and *e*, there is one separate part; in *f* there are two separate parts.

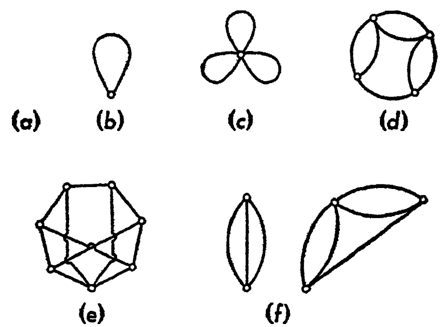


FIG. 2·4. Examples of geometric patterns. *a*, *b*, and *c* are of only theoretical interest.

One node in each separate part is chosen the reference node for that part. Each of the remaining nodes of any part taken in combination with the reference node of that part constitutes an independent node-pair. If k of these reference nodes are grounded, the number of nodes and the number of separate parts are each reduced by $k-1$, since grounding is equivalent to making the reference nodes coincide. (Fig. 2·5.)

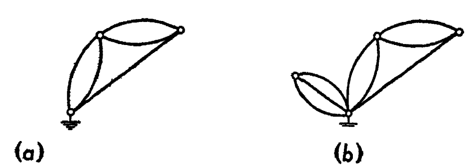


FIG. 2·5. Geometrically, *a* is the same as *b*.

The geometric patterns of electric networks are called linear graphs [Ki 1]. In the theory of linear graphs no distinction is made in the types of elements making up a branch, and the basic units are the branches. In the theory of linear electric networks, however, there are four kinds of elements, and electrically the behavior of each of these elements is distinctly different. Because of this, in discussing the geometry of linear electric networks each element is replaced by a line segment.

Two important general relations taken from the theory of linear

graphs [WH 3 to 6] and modified slightly so as to be immediately applicable to network geometry are [CA 1, Fo 1]:

$$l = e - n + s \tag{7}$$

$$n_p = n - s, \tag{8}$$

in which

$l \triangleq$ number of independent geometric loops,
$n_p \triangleq$ number of independent geometric node-pairs,
$e \triangleq$ number of elements,
$n \triangleq$ number of nodes, and
$s \triangleq$ number of separate parts.

Relations 7 and 8 can be illustrated by applying them to the geometric patterns of a variety of networks. For example, when applied to the pattern of Fig. 2·3 they give $l = 5$ and $n_p = 6$. Applied to the examples of Fig. 2·4, they give for

a	$l = 0$	$n_p = 1$
b	$l = 1$	$n_p = 0$
c	$l = 3$	$n_p = 0$
d	$l = 4$	$n_p = 3$
e	$l = 5$	$n_p = 6$
f	$l = 5$	$n_p = 3$

For the patterns a and b of Fig. 2·5 they give $l = 5$ and $n_p = 3$.

To describe the electrical behavior of a network a system of references must be established for the variables. For the instantaneous current in a branch an arrow is placed beside this branch to show the direction assumed for positive current. For the instantaneous voltage drop between the terminals of an element, branch, or network + and − signs are placed on these terminals to show the polarity assumed for positive voltage drop. The conventions here are the same as given above for current and voltage sources.

3. KIRCHHOFF'S LAWS

The differential equations of electric networks can be formulated directly by using Kirchhoff's voltage law and current law. The voltage law can be stated as follows:

Around any closed path in the network the sum of the instantaneous voltage drops in a specified direction is zero, or

$$\sum v_k(t) \text{ around a closed path} = 0. \tag{9}$$

The current law can be stated as follows:

> In the branches connected to a common node the sum of the instantaneous currents in the direction away from (or into) this node is zero, or
>
> $$\sum i_k(t) \text{ at a common node} = 0. \qquad [10]$$

These laws will be used now to write the integrodifferential equations for several simple electric networks.

4. ONE-LOOP *LRS* NETWORK

In the one-loop *LRS* network shown in Fig. 2·6 the switch K is closed at $t = 0$. Initially the circuit contains no stored energy. This means that the initial charge on the elastance and the initial current in the inductance are both zero.

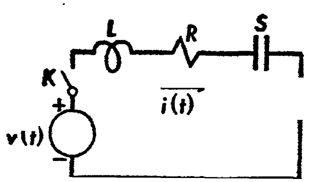


FIG. 2·6. One-loop *LRS* network.

In this circuit there are 3 unknowns — the current in the loop and the voltage drops between each of two nodes and a third node taken as a reference. There is one voltage source and no current source. Considered geometrically, the circuit has 1 loop and 3 independent node-pairs. By count, the number of unknown currents (1) equals the number of independent geometric loops (1) diminished by the number of current sources (0); and the number of unknown voltage drops to reference (2) equals the number of independent geometric node-pairs (3) diminished by the number of voltage sources (1).

If all three unknowns are to be found, three equations will be necessary. If only the current is wanted, one equation will suffice, whereas, if only one voltage drop is wanted, two equations will be necessary (either one current and one voltage-drop equation, or two voltage-drop equations).

Let it be required here to find only the current. The current $i(t)$ can be taken as the single dependent variable. The arrow direction, i.e., the direction for positive current, is taken as clockwise in the loop. By Kirchhoff's voltage law, the sum of the instantaneous voltage drops in the positive direction is zero, i.e.,

$$v_L(t) + v_R(t) + v_S(t) - v(t) = 0.$$

Substituting the expressions for these drops,

$$L\frac{di}{dt} + Ri + S\int i\,dt = v(t), \qquad i \triangleq i(t). \qquad [11]$$

As a consequence of the zero initial voltage across S,

$$S \int i dt = S \int_0^t i dt$$

and equation 11 can be written

$$L \frac{di}{dt} + Ri + S \int_0^t i dt = v(t). \qquad [12]$$

In this way one initial condition is included in the i-d equation. The remaining initial condition is $i(0) = 0$.

5. Two-Loop Network with Coupling through Elastance

In the two-loop network of Fig. 2·7–a switch K is closed at $t = 0$. The initial voltage across S_1 is zero, but as a result of a previous switching operation there is an initial voltage of magnitude γ across S_2 and an initial current of magnitude ρ in L_1. The polarity of this initial voltage and the direction of this initial current are shown on the diagram. The i-d equations for this network will now be formulated.

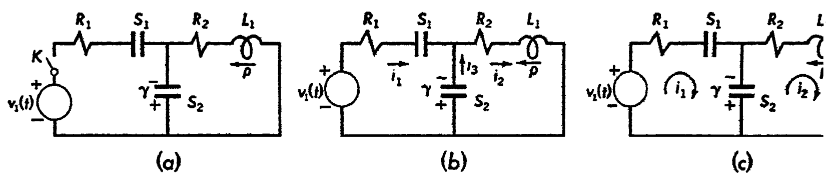


FIG. 2·7. Indication of initial conditions, branch currents, and loop currents.

In specifying the initial conditions of a system it is helpful to keep separate the ideas of magnitude and sign. In particular, letters such as γ, ρ, λ will be used to indicate only the magnitude of the initial value, and the sign will be written explicitly. For example, $v(0) = -\gamma$, and $i(0) = -\rho$.

Let the branch currents i_1, i_2, and i_3 and the arrow direction for each be chosen as indicated in Fig. 2·7–b. By Kirchhoff's current law,

$$-i_1 + i_2 - i_3 = 0. \qquad [13]$$

But from equation 13 it is seen that there need be only two dependent variables, e.g., currents i_1 and i_2, since $i_3 = i_2 - i_1$. Currents i_1 and i_2 can be considered to be localized in loops 1 and 2, respectively, as shown in Fig. 2·7–c. A current designated for an entire loop in this manner is called a loop current. The arrow direction for a loop current is usually taken clockwise.

In this network there are 5 unknowns — the two loop currents, and the voltage drops between each of three nodes and a fourth node taken as reference. Geometrically, the network has 2 independent loops and 4 independent node-pairs. Since there is one voltage source and no current source, it is seen that the number of unknown loop currents (2) equals the number of independent geometric loops (2) diminished by the number of current sources (0); likewise the number of unknown node-pair voltage drops (3) equals the number of independent geometric node-pairs (4) diminished by the number of voltage sources (1). If the loop currents are chosen as the dependent variables, two simultaneous i-d equations will suffice to determine the network behavior, whereas, if the node-pair voltages are chosen, three i-d equations will be necessary. The desirability of choosing the loop currents as variables is evident, and this will be done.

Applying Kirchhoff's voltage law first to loop 1 and then to loop 2 there is obtained

$$R_1 i_1 + S_1 \int i_1 dt + S_2 \int (i_1 - i_2) dt - v_1(t) = 0, \tag{14}$$

$$R_2 i_2 + L_1 \frac{di_2}{dt} + S_2 \int (i_2 - i_1) dt = 0. \tag{15}$$

The initial conditions for the two elastances result in

$$\left.\begin{aligned} S_1 \int i_1 dt &= S_1 \int_0^t i_1 dt, \\ S_2 \int (i_1 - i_2) dt &= S_2 \int_0^t (i_1 - i_2) dt - \gamma. \end{aligned}\right. \tag{16}$$

Rewriting equations 14 and 15 in symmetric form, and including 16,

$$\left.\begin{aligned} R_1 i_1 + (S_1 + S_2) \int_0^t i_1 dt - S_2 \int_0^t i_2 dt &= v_1(t) + \gamma, \\ -S_2 \int_0^t i_1 dt + L_1 \frac{di_2}{dt} + R_2 i_2 + S_2 \int_0^t i_2 dt &= -\gamma. \end{aligned}\right\} \tag{17}$$

Thus two of the initial conditions have been included in the i-d equation; the remaining initial condition is $i_2(0) = -\rho$.

The integral $\int_0^t i_1 dt$ represents the charge brought to both S_1 and S_2 by the loop current i_1; the integral $\int_0^t i_2 dt$ represents the charge brought

to S_2 by loop current i_2. When $t = 0$, these integrals are zero. The initial voltage across S_2 influences the subsequent currents in loop 1 like a battery whose magnitude is γ and whose polarity is such as to cause a positive i_1. It influences the subsequent currents in loop 2 like a battery whose magnitude is γ and whose polarity is such as to cause a negative i_2. It is seen, therefore, that equations 17 could have been written directly by reasoning physically.

6. Number of Dependent Variables Governed by Basis Used

In setting up the i-d equations for an electric network, it is usually desirable to choose either the unknown loop currents or the unknown node-pair voltage drops as the dependent variables since this leads to symmetry in the equations. In certain cases, however, the use of branch currents and branch voltage drops may be preferable to simplify the formulation of the equations.

If the loop currents are chosen as the dependent variables, the equations are said to be formed on the *loop basis*. The number of dependent variables will be equal to the number of independent geometric loops l except where there are current sources present. With current sources present the number of dependent variables needed is less than l by the number of these current sources, since each current source produces a known branch current. Thus, if

$u_i \triangleq$ number of unknown loop currents needed on the loop basis,
$l \triangleq$ number of independent geometric loops, equation 7, and
$i_s \triangleq$ number of current sources,

then

$$u_i = l - i_s. \tag{18}$$

If the node-pair voltage drops are chosen as the dependent variables, the equations are said to be formed on the *node basis*. The number of dependent variables will be equal to the number of independent geometric node-pairs n_p except where there are voltage sources present. With voltage sources present the number of dependent variables needed is less than n_p by the number of voltage sources, since each voltage source produces a known node-pair voltage or a known difference between two node-pair voltages. Thus, if

$u_v \triangleq$ number of unknown node-pair voltages needed on the node basis,
$n_p \triangleq$ number of independent geometric node-pairs, equation 8, and
$v_s \triangleq$ number of voltage sources,

then

$$u_v = n_p - v_s. \tag{19}$$

In choosing a basis upon which to formulate the network equations, that basis is selected which leads most directly to the result desired and requires the least number of unknowns and equations. So far only the loop basis has been used, but the node basis will be presented later in this chapter. In the two networks already considered, the number of unknown loop currents was less than the number of unknown node-pair voltages, so the loop basis was the logical one to use.

7. Two-Loop Network with Coupling through Resistance and Inductance

Two circuits carrying currents can store energy mutually in the magnetic field. This energy may be divided into three parts — one accounted for by the self-inductance L_1 of circuit 1, a second accounted for by the self-inductance L_2 of circuit 2, and a third accounted for by the mutual inductance M between circuits 1 and 2. Mutual inductance is represented diagrammatically as shown in Fig. 2·8. Although a symbol for mutual inductance is introduced, mutual inductance will not be considered to be a network element.

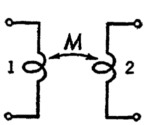


FIG. 2·8. Representation of mutual inductance.

If the current in circuit 1 is varying, the open-circuit voltage between terminals in circuit 2 is given by

$$v_2(t) = M\frac{di_1(t)}{dt}.$$

Similarly, if the current in circuit 2 is varying, the open-circuit voltage in circuit 1 is given by

$$v_1(t) = M\frac{di_2(t)}{dt}.$$

M will always be a positive real number. The signs to give the $M\dfrac{di}{dt}$ terms in the i-d equations will be discussed below.

The two loops of the network shown in Fig. 2·9 are coupled conductively by the common resistance R_3 and self-inductance L_3 and coupled inductively by the mutual inductance M. The switch K is closed at $t = 0$. The initial current in L_1 is zero, but there is an initial current of magnitude ρ in L_2 and L_3. The direction of this initial current is shown on the diagram. The differential equations for this network will now be formulated.

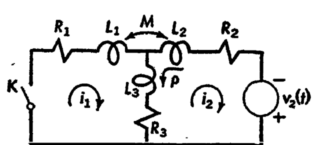


FIG. 2·9. Two-loop network with coupling through R_3, L_3, and M.

This network has 2 independent geometric loops and 5 independent geometric node-pairs. Since the number (see Sec. 6) of loop variables needed is 2, whereas the number of node variables needed is 4, the loop currents i_1 and i_2 are chosen as the dependent variables, with positive directions as indicated in the diagram.

Kirchhoff's voltage law applied first to loop 1 and then to loop 2 gives

$$
\begin{aligned}
&L_1 \frac{di_1}{dt} + R_1 i_1 + L_3 \frac{d}{dt}(i_1 - i_2) + R_3(i_1 - i_2) \pm M \frac{di_2}{dt} = 0, \\
&L_2 \frac{di_2}{dt} + R_2 i_2 + L_3 \frac{d}{dt}(i_2 - i_1) + R_3(i_2 - i_1) \pm M \frac{di_1}{dt} - v_2(t) = 0,
\end{aligned}
\tag{20}
$$

in which the signs for the mutual-induction terms are left undetermined for the present. With the terms arranged symmetrically, equations 20 become

$$
\begin{aligned}
&(L_1 + L_3)\frac{di_1}{dt} + (R_1 + R_3)i_1 - (L_3 \mp M)\frac{di_2}{dt} - R_3 i_2 = 0, \\
&-(L_3 \mp M)\frac{di_1}{dt} - R_3 i_1 + (L_2 + L_3)\frac{di_2}{dt} + (R_2 + R_3)i_2 = v_2(t).
\end{aligned}
\tag{21}
$$

The initial conditions are

$$
\begin{aligned}
i_1(0) &= 0, \\
i_2(0) &= -\rho.
\end{aligned}
$$

With these values the initial current in L_1 is zero, the initial current in L_2 is $-\rho$, and the initial current in L_3 is $[i_1(0) - i_2(0)] = \rho$, all of which are in agreement with the problem statement.

Returning now to the matter of signs in equations 20, the following question for the $M \dfrac{di_2}{dt}$ term must be answered: Does a positive rate-of-change of i_2 induce in loop 1 a voltage drop or a voltage rise in the arrow direction? If it is a drop, the sign of the term is +, if a rise it is −. These signs are in accordance with the convention that was used in writing the equations, i.e., voltage drops in the arrow direction are considered positive.

A similar question for the $M \dfrac{di_1}{dt}$ term must be answered: Does a positive rate-of-change of i_1 induce in loop 2 a voltage drop or a voltage rise in the arrow direction? If it is a drop, the sign is +.

To answer these questions certain additional information must be

available regarding the mutually coupled coils. There are at least two ways in which this information can be given.

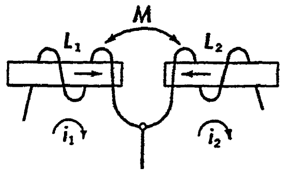


FIG. 2·10. Coils L_1 and L_2 of Fig. 2·9 shown wound on cores to display their winding senses.

a. The winding senses of the two coils can be shown. Suppose the coils are wound as indicated in Fig. 2·10. The arrow directions for loop currents i_1 and i_2 are shown, also the corresponding arrow directions for the magnetic field intensities along the axes of the coils. If $\frac{di_2}{dt}$ is positive, the associated induced voltage in loop 1 is a rise in the arrow direction. The sign of the $M\frac{di_2}{dt}$ term in equation 20 is accordingly $-$. Similarly, if $\frac{di_1}{dt}$ is positive, the associated induced voltage in loop 2 is a rise in the arrow direction. The sign of the $M\frac{di_1}{dt}$ term is likewise $-$. The pairing of signs in this way is characteristic of mutual-induction terms.

b. The second way in which the necessary additional information can be given is by indication of the polarity of the induced voltage in each coil resulting from positive rate-of-change of current in one of the coils. The coil not carrying current is assumed to be open-circuited at its terminals. For the configuration shown in Fig. 2·10, the $\pm$ signs on the coil terminals would be as shown in Fig. 2·11–*a*.

If the second way is to be used, the $\pm$ signs to be placed on the coil terminals can be determined experimentally by disconnecting the coils from the network but leaving them unchanged in position relative to each other. A d-c voltmeter is connected across the terminals of one

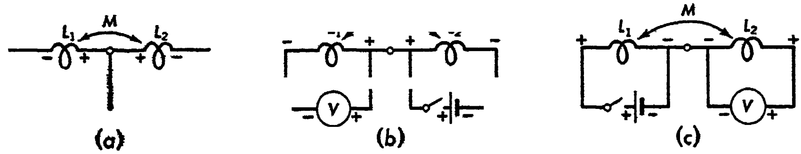


FIG. 2·11. Terminal polarity markings can be established by test with battery and voltmeter.

coil, and a battery is connected to the terminals of the other coil as in Fig. 2·11–*b*. Upon closing the battery circuit the voltmeter will give a

momentary indication. If this indication is upscale, the terminal of the coil connected to the positive terminal of the voltmeter should be marked +; if the indication is downscale, the terminal of the coil connected to the negative terminal of the voltmeter should be marked +. The corresponding terminals on the exciting coil are marked to agree with the battery polarity. The battery tends to set up a current in the coil to which it is connected in the direction of the voltage drop through the coil. The build-up of this current causes a potential difference to appear at the terminals of the coil connected to the voltmeter. It can readily be demonstrated that a similar test made with the voltmeter and battery interchanged (Fig. 2·11–*c*) gives consistent information. For this reason a single test is sufficient.

This simple polarity test is easily carried out. It is practically indispensable when the coupling occurs in irregular and incalculable coil or circuit configurations. When three or more coils are mutually coupled, one test may be sufficient, but in general with n coils, all mutually coupled, $n - 1$ separate tests are needed to establish the pairs of signs to be associated with each of the $n(n - 1)/2$ mutual inductances.

Having the inductively coupled coils marked in this way and connected in the network, and the arrow directions in the loops assigned, it is an easy matter to establish the signs of the terms in the differential equations. This is shown in the example in the following section.

8. Three-Loop Network

In the 3-loop network of Fig. 2·12 there is mutual induction between coils 1 and 2, 1 and 3, and 2 and 3. The switch K is closed at $t = 0$. At this instant the currents in the inductances have the magnitudes and directions indicated, and the voltage drops across the elastances have the magnitudes and polarities indicated. The i-d equations for this system will now be formulated.

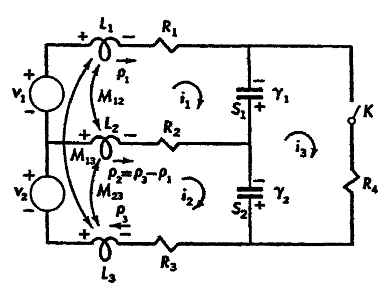


FIG. 2·12. Network with three coils inductively coupled.

The network has 3 independent geometric loops and 8 independent geometric node-pairs. The number of loop variables needed is 3, whereas the number of node variables needed is 6; consequently the three loop currents i_1, i_2, and i_3, with positive directions as shown in the diagram, are chosen as the dependent variables.

The i-d equations are

$$\left.\begin{aligned}
&L_1 \frac{di_1}{dt} + R_1 i_1 + S_1 \int_0^t (i_1 - i_3)dt - \gamma_1 + R_2(i_1 - i_2) \\
&\qquad + L_2 \frac{d}{dt}(i_1 - i_2) - M_{21}\frac{di_1}{dt} - M_{12}\frac{d}{dt}(i_1 - i_2) \\
&\qquad - M_{13}\frac{di_2}{dt} + M_{23}\frac{di_2}{dt} = v_1, \\
&L_2 \frac{d}{dt}(i_2 - i_1) + R_2(i_2 - i_1) + S_2 \int_0^t (i_2 - i_3)dt \\
&\qquad - \gamma_2 + R_3 i_2 + L_3 \frac{di_2}{dt} - M_{31}\frac{di_1}{dt} - M_{32}\frac{d}{dt}(i_2 - i_1) \\
&\qquad + M_{21}\frac{di_1}{dt} - M_{23}\frac{di_2}{dt} = v_2, \\
&S_1 \int_0^t (i_3 - i_1)dt + \gamma_1 + R_4 i_3 + S_2 \int_0^t (i_3 - i_2)dt + \gamma_2 = 0.
\end{aligned}\right\} \quad [22]$$

The signs for the mutual-induction terms are determined by the $\pm$ signs on the coils in accordance with the convention presented in Sec. 7.

Two initial conditions, those for the elastance voltages, have been included in the equations. The necessary additional initial conditions for the inductance currents are

$$i_1(0) = \rho_1, \qquad i_2(0) = \rho_3.$$

9. The l-Loop Network

It will be noted that as the complexity of the network increases it becomes more cumbersome to write out the i-d equations in full. The need for a briefer notation is evident. The following scheme serves effectively to condense the equations and systematize the calculations.

With the loops projected on a plane, choose the arrow directions of the loop currents all clockwise. Let

$L_{jj} \triangleq$ total self-inductance on loop j; this may include mutual inductance between elements on this loop.
$R_{jj} \triangleq$ total resistance on loop j,
$S_{jj} \triangleq$ total elastance on loop j,
$L_{jk} \triangleq$ total self-inductance and mutual inductance common to loops j and k,
$R_{jk} \triangleq$ total resistance common to loops j and k,
$S_{jk} \triangleq$ total elastance common to loops j and k.

A self i-d operator for loop j can be defined as

$$a_{jj} \triangleq L_{jj}\frac{d}{dt} + R_{jj} + S_{jj}\int dt,$$

and a mutual i-d operator for loops j and k as

$$a_{jk} \triangleq -L_{jk}\frac{d}{dt} - R_{jk} - S_{jk}\int dt.$$

Certain exceptions may be mentioned. As the result of coupling between loops through mutual inductance, a positive derivative term in the mutual i-d operator may be unavoidable. Also, in a network whose geometric pattern cannot be mapped on a plane without crossing of branches, mutual terms that are positive will occur because certain branches must be taken as common to more than two loops.

Assuming that all the sources present are voltage sources, and letting the $i(t)$'s represent the currents in the l independent loops and the $v(t)$'s the net driving voltages in these loops, the i-d equations for the network can be written

$$\left.\begin{array}{l} a_{11}i_1(t) + a_{12}i_2(t) + \cdots + a_{1l}i_l(t) = v_1(t), \\ a_{21}i_1(t) + a_{22}i_2(t) + \cdots + a_{2l}i_l(t) = v_2(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{l1}i_1(t) + a_{l2}i_2(t) + \cdots + a_{ll}i_l(t) = v_l(t). \end{array}\right\} \qquad [23]$$

This set of equations can be written more compactly as

$$\sum_{k=1}^{l} a_{jk}i_k(t) = v_j(t), \qquad j = 1, 2, \cdots, l. \qquad [24]$$

The index j designates the loop and also the equation for that loop in the set 23. The index k designates the loop current whose effect in loop j is being considered; it also indicates the particular term in the jth equation.

Although the brevity of statement presented in equation 24 is useful in formulating general network relations, it is not useful in numerical solutions until expanded as in equations 23.

As an example of the application of this notation, consider the network shown in Fig. 2·9, the equations for which are given in 21. The i-d equations 21 are

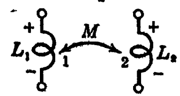
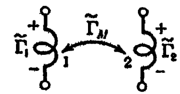
$$a_{11}i_1(t) + a_{12}i_2(t) = 0,$$

$$a_{21}i_1(t) + a_{22}i_2(t) = v_2(t).$$

TABLE 1. ELECTRIC NETWORKS

COMPARISON OF LOOP AND NODE BASES OF ANALYSIS

	Loop Basis		Diagrammatic Symbol	Node Basis	
1	$-\ \ +$ Indicates polarity when $v(t)$ has a positive value.	$-\ \ +$ $v(t)$ Voltage	 Source	$\vec{i}(t)$ Current	$\rightarrow$ Indicates direction of current when $i(t)$ has a positive value.
2	$v_R = Ri_R$	R Resistance	 Sink (heat)	G Conductance	$i_G = Gv_G$ $G \triangleq R^{-1}$
	$v_S = S \int i_S dt$	S Elastance	 Reservoir (electric)	C Capacitance	$i_C = C \dfrac{dv_C}{dt}$ $C \triangleq S^{-1}$
	$v_L = L \dfrac{di_L}{dt}$	L Self-inductance	 Reservoir (magnetic) Self-inductance (only)	Γ Inverse self-inductance	$i_\Gamma = \Gamma \int v_\Gamma dt$ $\Gamma \triangleq L^{-1}$

3	—	L_1 Self-inductance of 1 (2 open) L_2 Self-inductance of 2 (1 open) M Mutual inductance	For loop basis  For node basis  Reservoir (magnetic) Self-inductance and mutual inductance.	$\tilde{\Gamma}_1$ Inverse self-inductance of 1 (2 short-circuited) $\tilde{\Gamma}_2$ Inverse self-inductance of 2 (1 short-circuited) $\tilde{\Gamma}_M$ Inverse mutual inductance.	$\tilde{\Gamma}_1 \triangleq \dfrac{L_2}{L_1L_2 - M^2}$ $\tilde{\Gamma}_2 \triangleq \dfrac{L_1}{L_1L_2 - M^2}$ $\tilde{\Gamma}_M \triangleq \dfrac{M}{L_1L_2 - M^2}$
4	—	$l = e - n + s$ Number of independent geometric loops.	—	$n_p = n - s$ Number of independent geometric node-pairs.	Ground node is reference.
5	Number of loop equations needed.	$u_i = l - i_{\text{sources}}$ Number of unknown loop currents.	—	$u_v = n_p - v_{\text{sources}}$ Number of unknown node-pair voltages.	Number of node equations needed.
6	Voltage drop in direction of traverse around loop is usually taken as positive.	Around a closed path $\sum_{k=1}^{l} v_k(t) = 0$ Kirchhoff's voltage law	—	At a common node $\sum_{k=1}^{n_p} i_k(t) = 0$ Kirchhoff's current law.	Currents directed away from node are usually taken as positive.

Explanation of Divisions of Table:
1. Active elements.
2. Passive elements.
3. Mutual inductance.
4. Independent loops and node-pairs in geometric pattern of network.
5. Number of dependent variables needed.
6. Law of electric circuits used in writing equations.

The i-d operators are

$$a_{11} \triangleq L_{11}\frac{d}{dt} + R_{11},$$

$$a_{22} \triangleq L_{22}\frac{d}{dt} + R_{22},$$

$$a_{12} = a_{21} \triangleq -L_{12}\frac{d}{dt} - R_{12}.$$

Recalling that the signs of the $M\dfrac{di}{dt}$ terms in equations 20 were subsequently shown to be $-$, the loop and mutual constants for the network are

$$L_{11} \triangleq L_1 + L_3, \qquad L_{22} \triangleq L_2 + L_3, \qquad L_{12} = L_{21} \triangleq L_3 + M,$$

$$R_{11} \triangleq R_1 + R_3, \qquad R_{22} \triangleq R_2 + R_3, \qquad R_{12} = R_{21} \triangleq R_3.$$

10. Node Basis of Formulating Network Equations

So far the examples of network equations have been written on the loop basis and have been founded on Kirchhoff's voltage law. The equations for a network can likewise be written on the node basis, i.e., can be founded on Kirchhoff's current law. In the loop scheme attention is directed to the independent loop currents of the network; in the node scheme attention is directed to the independent node-pair voltages of the network [Mi 2, Ru 1]. The latter is the less familiar scheme but is the simpler and more direct one to use when the number of unknown node-pair voltages is less than the number of unknown loop currents. A comparison of the loop and node bases of analysis is made in Table 1. Explanation of the inverse inductances included in Division 3 of this table is given below in Sec. 13.

In the node scheme, one node of each separate part of the network is taken as the reference node for that part. The potential, with respect to this reference, of each other node in that part becomes a node-pair voltage, and is indicated at the node by a $v(t)$ with identifying subscript. Unless this node-pair voltage happens to be a known-source voltage it becomes one of the dependent variables. A system of reference for the dependent variables is established by marking the reference nodes $-$ and all other nodes $+$. At any instant at which a node-pair voltage drop has a positive numerical value, the corresponding node has a positive potential with respect to the reference.

11. ONE-NODE-PAIR NETWORK

In the network of Fig. 2·13, the opening of the switch K at $t = 0$ impresses a known current $i(t)$ upon the $CG\Gamma$-parallel group. The initial voltage across C and the initial current in Γ are both zero. The i-d equation for the network will be written.

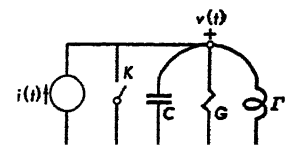


FIG. 2·13. One-node-pair network.

This network has 3 independent geometric loops and 1 independent geometric node-pair. Since the number of unknown loop currents is 2, whereas the number of unknown node-pair voltages is 1, the single node-pair voltage drop $v(t)$ to reference will be taken as the dependent variable, and the equation for the network will be written on the node basis.

By Kirchhoff's current law, the sum of the instantaneous currents in the direction away from the node is zero, i.e.,

$$i_C(t) + i_G(t) + i_\Gamma(t) - i(t) = 0. \qquad [25]$$

Substituting the expressions for these currents,

$$C\frac{dv}{dt} + Gv + \Gamma\int vdt = i(t), \qquad v \triangleq v(t). \qquad [26]$$

Since the initial current in Γ is zero,

$$\int vdt = \Gamma\int_0^t vdt,$$

and equation 26 can be written

$$C\frac{dv}{dt} + Gv + \Gamma\int_0^t vdt = i(t). \qquad [27]$$

In this way one initial condition is included in the i-d equation. The remaining initial condition is $v(0) = 0$.

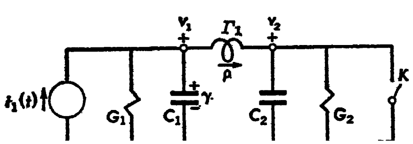


FIG. 2·14. Two-node-pair network.

12. TWO-NODE-PAIR NETWORK

The switch K in the network of Fig. 2·14 is opened at $t = 0$. At this instant the voltage across C_1 has a magnitude γ, and the current in Γ_1 a magnitude ρ. The polarity of this initial voltage and the direction of this initial current are shown on the diagram. The i-d equations for the network will be formulated.

Viewed geometrically, this network has 4 independent loops and 2 independent node-pairs. There are 3 unknown loop currents, but only 2 unknown node-pair voltages, consequently the node voltage drops v_1 and v_2 are chosen as the dependent variables.

Using Kirchhoff's current law, one equation is written for node 1, and a second is written for node 2.

$$\left.\begin{aligned} G_1v_1 + C_1\frac{dv_1}{dt} + \Gamma_1\int (v_1 - v_2)dt - i_1(t) &= 0, \\ G_2v_2 + C_2\frac{dv_2}{dt} + \Gamma_1\int (v_2 - v_1)dt &= 0. \end{aligned}\right\} \quad [28]$$

Because of the initial current in Γ_1,

$$\Gamma_1\int (v_1 - v_2)dt = \Gamma_1\int_0^t (v_1 - v_2)dt + \rho. \quad [29]$$

When relation 29 is included in equations 28, and they are written in symmetric form,

$$\left.\begin{aligned} C_1\frac{dv_1}{dt} + G_1v_1 + \Gamma_1\int_0^t v_1dt - \Gamma_1\int_0^t v_2dt &= i_1(t) - \rho, \\ -\Gamma_1\int_0^t v_1dt + C_2\frac{dv_2}{dt} + G_2v_2 + \Gamma_1\int_0^t v_2dt &= \rho. \end{aligned}\right\} \quad [30]$$

One of the initial conditions is thus included in the i-d equations; the other two initial conditions are $v_1(0) = \gamma$ and $v_2(0) = 0$.

13. Inverse Inductances with Mutual Induction Present

When the i-d equations for a network containing mutual inductances are to be written on the node basis, the self-inductances and mutual inductances are replaced by inverse self-inductances and inverse mutual inductances. Since these inverse inductances are not simply the reciprocals of the self-inductances and mutual inductances, they require special comment.

In contrast to self-inductance and mutual inductance, which are open-circuit constants, inverse self-inductance and inverse mutual inductance are short-circuit constants. The *inverse self-inductance* of an inductive element is defined by the ratio of the current in this element to the integral of the voltage drop across its terminals — all other elements to which it is inductively coupled being short-circuited. These short-circuited elements will in general carry currents. The *inverse mutual*

inductance of two inductively coupled elements is defined by the ratio of the short-circuit current in one of these elements to the integral of the voltage drop across the other element — all inductively coupled elements except the latter being short-circuited. The diagrammatic representations for inverse self-inductance and inverse mutual inductance are the same respectively as for self-inductance and mutual inductance.

The inverse inductances can be expressed in terms of the self-inductances and mutual inductances. The relations among them can be derived from the voltage equations for the coupled elements by solving these equations for the element currents in terms of the node voltages at the terminals of the elements.

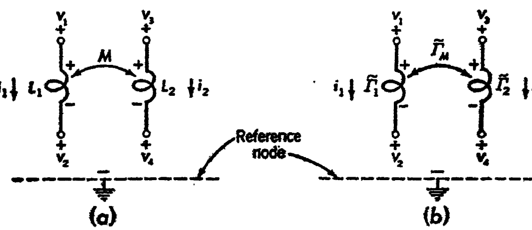


FIG. 2·15. Inductively coupled elements. Form given in a better for analysis on loop basis; form given in b better for node basis.

As an example of this procedure, the inverse inductances for two coupled elements (Fig. 2.15–a) will be found. Assume that these elements are part of a larger network of which only the reference node is shown; assume also that they are the only inductances connected to nodes 1, 2, 3, and 4 and that they are not inductively coupled to other elements in the network. At their terminals the node-pair voltage drops to reference are v_1, v_2, v_3, and v_4. Let the currents i_1 and i_2 in these elements have the arrow directions shown. Then the voltage equations for these elements are

$$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = v_1 - v_2, \qquad [31]$$

$$M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = v_3 - v_4. \qquad [32]$$

Eliminating $\frac{di_2}{dt}$ from equations 31 and 32,

$$(L_1L_2 - M^2) \frac{di_1}{dt} = L_2(v_1 - v_2) - M(v_3 - v_4). \qquad [33]$$

Eliminating $\dfrac{di_1}{dt}$ from equations 31 and 32,

$$(L_1L_2 - M^2)\frac{di_2}{dt} = -M(v_1 - v_2) + L_1(v_3 - v_4). \qquad [34]$$

Integration of equations 33 and 34 gives

$$\begin{aligned} i_1 &= \tilde{\Gamma}_1 \int (v_1 - v_2)dt - \tilde{\Gamma}_M \int (v_3 - v_4)dt \\ &= \tilde{\Gamma}_1 \int_0^t (v_1 - v_2)dt - \tilde{\Gamma}_M \int_0^t (v_3 - v_4)dt + i_1(0), \end{aligned} \qquad [35]$$

$$\begin{aligned} i_2 &= -\tilde{\Gamma}_M \int (v_1 - v_2)dt + \tilde{\Gamma}_2 \int (v_3 - v_4)dt \\ &= -\tilde{\Gamma}_M \int_0^t (v_1 - v_2)dt + \tilde{\Gamma}_2 \int_0^t (v_3 - v_4)dt + i_2(0), \end{aligned} \qquad [36]$$

in which $\tilde{\Gamma}_1 \triangleq \dfrac{L_2}{L_1L_2 - M^2}$ is the inverse self-inductance of element 1, with 2 short-circuited,

$\tilde{\Gamma}_2 \triangleq \dfrac{L_1}{L_1L_2 - M^2}$ is the inverse self-inductance of element 2, with 1 short-circuited, and

$\tilde{\Gamma}_M \triangleq \dfrac{M}{L_1L_2 - M^2}$ is the inverse mutual inductance between elements 1 and 2, with one element short-circuited.

The $\sim$ (tilde) is added to indicate that $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, and $\tilde{\Gamma}_M$ are not simply the reciprocals of L_1, L_2, and L_M, respectively. When the inverse inductances are substituted in the diagram it becomes that shown in Fig. 2·15–*b*.

Currents i_1 and i_2, for which expressions are given in equations 35 and 36, are the total inverse-inductance currents directed away from nodes 1 and 3, respectively and are the total inverse-inductance currents directed toward nodes 2 and 4, respectively. By inclusion of $i_1(0)$ and $i_2(0)$ in equations 35 and 36, provision is made for any initial currents.

$\tilde{\Gamma}_1 \int (v_1 - v_2)dt$ is the self-component of i_1. It is directed away from node 1 and is $+$.

$\tilde{\Gamma}_M \int (v_3 - v_4)dt$ is the induced component of i_1. With the $\pm$ signs as in Fig. 2·15–*b*, this induced component is directed toward node 1 and hence is $-$.

$\tilde{\Gamma}_2 \int (v_3 - v_4)dt$ is the self-component of i_2. It is directed away from node 3 and is +.

$\tilde{\Gamma}_M \int (v_1 - v_2)dt$ is the induced component of i_2. With the $\pm$ signs as in Fig. 2·15–*b* this induced component is directed toward node 3 and hence is −.

A similar procedure can be followed if there are three or more inductively coupled elements. This transformation to inverse inductances is a necessary preliminary to writing the network equations on the node basis.

14. EXCHANGE OF SOURCES

The i-d equations written on the node basis are current equations. When formulating them it is convenient to have only current sources. When there are voltage sources present in series with passive elements, these series combinations can be replaced in the network diagram by parallel combinations of current sources and passive elements that will maintain all terminal and initial conditions invariant. This is an analytic device, used solely for convenience in writing the network equations. A similar exchange of current for voltage sources can be made when the equations are to be formulated on the loop basis. The conditions to be fulfilled when these exchanges are to be made are given below.

A voltage source $v(t)$ in series with a resistance R (Fig. 2·16) can be replaced by a current source $i(t)$ in parallel with a conductance G and still maintain the terminal conditions invariant provided $i(t) = Gv(t)$, with $G = R^{-1}$. In justification of this, let $v_1(t)$ and $i_1(t)$ be respectively the terminal voltage drop and terminal current that are to be kept invariant; then for the series connection,

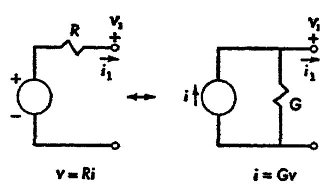


FIG. 2·16. Exchange of sources when the series element is resistance. $G = R^{-1}$.

$$
\begin{aligned}
i_1 &= \frac{1}{R}(v - v_1) \\
&= Gv - Gv_1 \\
&= i - Gv_1, \qquad \text{with } i \triangleq Gv. \qquad [37]
\end{aligned}
$$

The last form of the equation can be represented by the parallel connection shown. Conversely, if a current source $i(t)$ in parallel with G is to

be replaced by a voltage source $v(t)$ in series with R, the voltage source must have the form $v(t) = Ri(t)$, with $R = G^{-1}$.

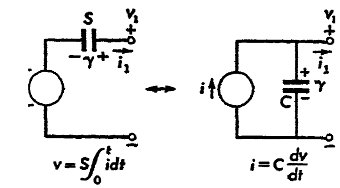


FIG. 2·17. Exchange of sources when the series element is elastance. $C = S^{-1}$.

Similarly, a voltage source $v(t)$ in series with an elastance S (Fig. 2·17) can be replaced by a current source $i(t)$ in parallel with a capacitance C provided $i(t) = C\dfrac{dv(t)}{dt}$, with $C = S^{-1}$. If there is an initial voltage drop across S of magnitude γ and polarity as shown, and $v(0) = 0$, the same initial voltage and polarity are retained across C. This exchange of sources can be justified as follows:

$$\begin{aligned} i_1 &= \frac{1}{S}\frac{d}{dt}(v + \gamma - v_1) \\ &= C\frac{dv}{dt} - C\frac{dv_1}{dt} \\ &= i - C\frac{dv_1}{dt}, \qquad \text{with } i \triangleq C\frac{dv}{dt}. \end{aligned} \tag{38}$$

For the corresponding replacement of a current source $i(t)$ by a voltage source $v(t)$, the voltage source must have the form $v(t) = S\displaystyle\int_0^t i(t)dt$.

Likewise, a voltage source $v(t)$ in series with an inductance L (Fig. 2·18) can be replaced by a current source in parallel with an inverse inductance Γ provided $i(t) = \Gamma\displaystyle\int_0^t v(t)dt$. If there is an initial current in L of magnitude ρ and direction as shown, this same initial current is retained in Γ. In justification of this,

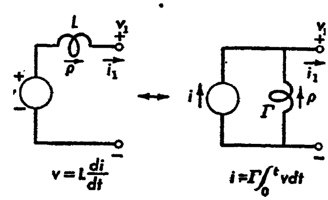


FIG. 2·18. Exchange of sources when the series element is inductance. $\Gamma = L^{-1}$.

$$\begin{aligned} i_1 &= \frac{1}{L}\int (v - v_1)dt \\ &= \frac{1}{L}\int_0^t (v - v_1)dt + \rho \\ &= \Gamma\int_0^t vdt - \Gamma\int_0^t v_1dt + \rho \\ &= i - \Gamma\int_0^t v_1dt + \rho, \qquad \text{with } i \triangleq \Gamma\int_0^t vdt. \end{aligned} \tag{39}$$

For the corresponding replacement of a current source $i(t)$ by a voltage

source $v(t)$, the voltage source must have the form $v(t) = L\dfrac{di(t)}{dt}$.

Each replacement of a voltage source in series with a passive element by a current source in parallel with the inverse of this element reduces by one the number of independent geometric node-pairs of the network. Similarly, each replacement in the opposite way reduces by one the number of independent geometric loops.

In Sec. 6 rules were given for determining the number of dependent variables necessary for treatment of a network on the node basis and on the loop basis. The number of unknowns for the node basis was less than the number of independent geometric node-pairs by the number of voltage sources present. Similarly the number of unknowns for the loop basis was less than the number of independent geometric loops by the number of current sources present. It is seen now that this reduction in the number of unknowns in either treatment is the reduction that can be made by an exchange of sources.

Example 1. In the network shown in Fig. 2·19–*a* the switch K is closed at $t = 0$. At that instant there is current in L of magnitude ρ in the direction indicated. The condenser is initially uncharged. The equation for the subsequent condenser voltage is wanted.

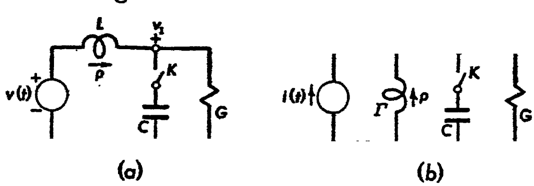


FIG. 2·19. A preliminary change of source. $i(t) = \Gamma\int_0^t v(t)dt$, with $\Gamma = L^{-1}$.

The number of unknown loop currents is 2, and of unknown node-pair voltages is 1, so it is convenient to treat the network on the node basis, especially since the node voltage and the wanted condenser voltage are the same. The voltage source in series with L is replaced by the current source in parallel with Γ. The initial current in Γ has the same magnitude and direction as before (Fig. 2·19–*b*). With $\Gamma = L^{-1}$ and $i(t) = \Gamma\int_0^t v(t)dt$, the i-d equation for the condenser voltage is

$$C\frac{dv_1}{dt} + Gv_1 + \Gamma\int v_1 dt = i(t),$$

or if the initial condition in Γ is written explicitly,

$$C\frac{dv_1}{dt} + Gv_1 + \Gamma\int_0^t v_1 dt - \rho = i(t).$$

The second initial condition is $v_1(0) = 0$, since there is no initial charge on C.

In discussing the exchange of sources, series and parallel branches containing only one kind of passive element have been considered. The method can be extended to branches containing more than one kind of element and also to series-parallel groups of branches; but difficulties arise in the application of the method unless a way is available of solving the differential equations readily. For example, in Fig. 2·20, a current source in parallel with a series group is replaced by a voltage source in series with this same group. This follows from the relation

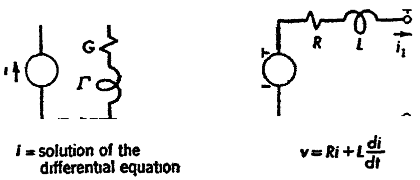


FIG. 2·20. Exchange of sources when the parallel branch contains two different kinds of elements.

$$v_1 = v_G + v_\Gamma = \frac{i - i_1}{G} + \frac{1}{\Gamma}\frac{d}{dt}(i - i_1)$$

$$= Ri + L\frac{di}{dt} - Ri_1 - L\frac{di_1}{dt}$$

$$= v - Ri_1 - L\frac{di_1}{dt}, \qquad \text{with} \quad v \triangleq Ri + L\frac{di}{dt}.$$

But the inverse of this process is not so easily carried out since i cannot be expressed in terms of v without solution of the differential equation

$$L\frac{di}{dt} + Ri = v.$$

Later, when differential equations can be $\mathfrak{L}$ transformed and algebraic methods can be used, the relation between i and v can be determined readily. In fact, groups of elements consisting of complicated series-parallel arrangements will present no serious difficulties, and an exchange of sources can be made freely whenever it is desirable.

15. ANALOGOUS OR DUAL NETWORKS

On comparison of the i-d equations 11 and 26 for the networks of Figs. 2·6 and 2·13, respectively, it will be noted that they have the same mathematical form. These two networks are thus two different physical representations of the same i-d equation.

Two systems that constitute two different physical representations of the same set of i-d equations are said to be *analogous systems*. Analogous electric networks are frequently called *duals* [Ru 1]. It is not necessary that the constants in the two systems be such as to give numerically equal coefficients in the two sets of i-d equations; it is sufficient if their equations have the same form.

It may be seen by comparison of the i-d equations of two electric networks that are duals, that the loop currents in one and the node-pair voltage drops in the other are analogous dependent variables. The equations for one network are sums of instantaneous voltage drops; for the dual network they are sums of instantaneous currents. Current sources are the duals of voltage sources, conductance is the dual of resistance, capacitance is the dual of inductance, and inverse inductance is the dual of elastance.

Not all networks have duals. Duality fails if the original network is nonplanar, i.e., if its geometric pattern can not be mapped on a plane without branches crossing.

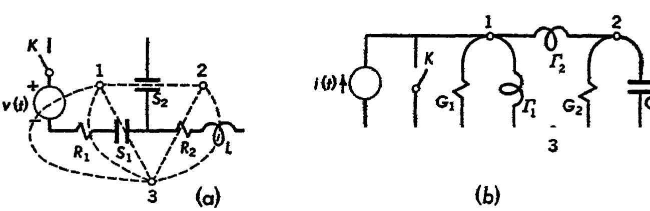


FIG. 2·21. Graphical construction to derive dual of a network.

If a dual network exists, a method of deriving it with certainty is to (1) formulate the i-d equations of the original network on either the loop or node basis, (2) rewrite these equations, inserting the duals of the variables, constants, and driving forces, and (3) interpret these derived equations by drawing the network whose behavior they characterize.

A convenient *graphical* scheme for finding the dual of a planar network is as follows. On the diagram of the network indicate one point within the enclosure of each independent loop, and also one outside the network. These points are to be the nodes of the dual network. The branches of the dual network are established by connecting these nodes. Between any two nodes insert one branch for each element common to the two loops enclosing these two nodes. Place in each such branch the dual of the element which it crosses.

As an example, the dual of the network discussed in Sec. 5 and shown in Fig. 2·21–*a* will be found.

Nodes 1, 2, and 3 are inserted as in Fig. 2·21–*a* and connected by lines

representing the branches of the dual network. The dual network is redrawn in Fig. 2·21–*b* with the dual elements replacing the line branches. That this derived network is the dual network can be verified by writing its i-d equations. They will have the same mathematical form as equations 14 and 15, but with different coefficients, dependent variables, and driving force.

If this process is repeated on the derived network the second derived network will be identical with the original network. In certain networks the first derived network is geometrically identical with the original network, i.e., the original network is topologically self-dual. For example, a bridge network having only single-element branches is self-dual.

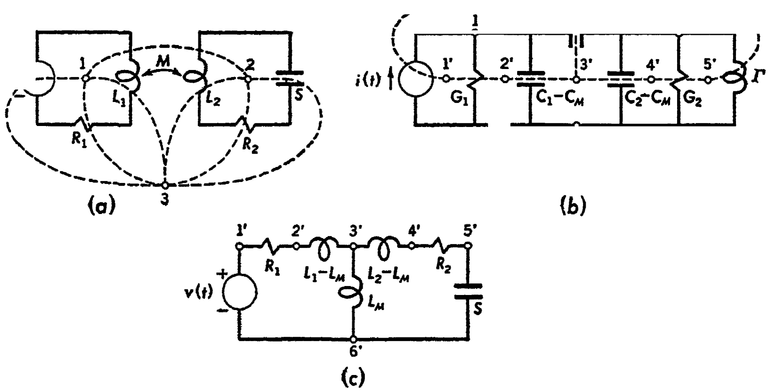


FIG. 2·22. Graphical construction to derive dual of a network containing mutual inductance.

Although mutual inductance has not been treated as an element, in the dual of a network that contains it a capacitance element corresponds to it. This introduces an exception to the principle of equality holding otherwise between the number of elements in the original and in the dual network. Furthermore, if subscripts 1 and 2 designate, respectively, the original and the dual network, then $l_1 = n_{p_2}$, but $n_{p_1} \neq l_2$. If subscript 3 designates the succeeding network derived from network 2, then $l_3 = n_{p_2} = l_1$, but $n_{p_3} = l_2 \neq n_{p_1}$. Network 3 may be recognizable as network 1 with all mutual inductances replaced by common self-inductances, and the essential compensating reductions made in the original self-inductances.

An example of the derivation graphically of the dual of a network in which mutual inductance is present is shown in Fig. 2·22, parts (*a*) and (*b*). The second derived network is shown in Fig. 2·22–*c*.

In a network that is nonplanar it is necessary for certain branches to serve as common parts of three or more loops. From the foregoing graphical scheme of deriving the dual of a network it can be seen that in any attempt to find a dual network a branch used in this way causes a serious geometric difficulty; its single dual branch should be common to three or more nodes—an evident geometric impossibility. If a network is planar then it has a topological dual; conversely if a network has a topological dual it is a planar network [WH 4]. Furthermore, it can be shown that if it is impossible for a dual to exist geometrically, it is likewise impossible for it to exist physically. For example, in the i-d equations for a nonplanar network, written on either the loop or node basis, the constants, variables, and sources may be replaced by their respective dual quantities; but the equations so derived, although of identical mathematical form with those for the original network, will not be representable by a physical network.

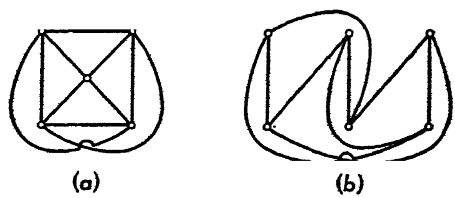


FIG. 2·23. Geometric patterns of the two simplest nonplanar networks.

The geometric patterns of the two simplest nonplanar networks are given in Fig. 2·23. Pattern *a* represents a 5-node network in which each node connects to each of the other 4 nodes. Pattern *b* represents a 6-node network in which each node of the upper group of three connects to each node of the lower group of three.

The distinction between the *optional* schemes for formulating the i-d equations for the *same* network, and the *duality* between two *different* networks, needs to be emphasized. The equations for a specified network may be written on either the loop or the node basis, that basis being selected which leads to the lesser number of equations or which is more convenient to use. This option is entirely independent of whether or not the specified network has a dual. If there is a dual it means that a second network exists whose i-d equations written on the node basis have the same form as the i-d equations for the original network written on the loop basis. Also it means that the equations for the dual written on the loop basis have the same form as the equations for the original network written on the node basis.

B. LUMPED-CONSTANT MECHANICAL SYSTEMS

Attention will be given here to certain principles of dynamics which will be helpful later in formulating the differential equations of motion

of rigid bodies subject to external constraints. By limiting the discussion to rigid bodies and to restraining elements of negligible mass, questions of bending and of wave propagation of stresses will be excluded. Only one independent variable (and for problems in transients this is time) will be necessary, thereby making the problems one dimensional. The equations of motion will be ordinary differential equations rather than partial differential equations. In brief, the systems will be considered to have lumped-constant rather than distributed-constant elements and to present " network " rather than field problems.

16. Coordinates of Motion

The motion of a rigid body can be one of translation, of rotation, or of combined translation and rotation. In translation, the path of each point of the body is parallel to the path of the center of gravity. In rotation one line of the body or an extension of the body remains fixed. Any general motion of the body can be regarded as a combination of translation and rotation, but if the lines about which the body rotates are considered to pass through its center of gravity, the translation and rotation are dynamically independent. The motion of any general point of the body can be expressed in terms of six coordinates: three translational coordinates such as x, y, and z for the center of gravity, and three rotational coordinates such as θ, ϕ, and ψ about orthogonal axes through the center of gravity.

Systems will be classified here according to the number of coordinates necessary to describe the motions of all their points. For example, a system in which a rigid body can move by translation only parallel to a plane and can rotate only about a line normal to this plane will be called a three-coordinate system. A spring-connected, three-body system in which the bodies can move only along the same straight line and can have no rotation will likewise be called a three-coordinate system. The coordinates or their time derivatives or integrals required to specify the motion become the dependent variables of the equations of motion.

17. Mechanical-System Elements

Lumped-constant mechanical systems can be considered as mechanical " networks." This simplifies the extension to mechanical systems of methods of analysis that have been developed for handling lumped-constant electric systems.

As with electric networks, mechanical networks are composed of active elements and passive elements. The active elements are the energy sources; the passive elements are the masses, springs, and mechanical resistors.

Consider first the translational system. There are two kinds of sources: force sources and velocity sources. A known driving force $f(t)$ acting in the system is representable in the system network as a force source. If instead of a force the driving velocity $v(t)$ of some point in the system is known, this known velocity is representable in the network as a velocity source applied at that point. Translational sources will be represented diagrammatically as shown in Fig. 2·24. The arrow on the source indicates the direction of the force when $f(t)$ has a positive value or the direction of the velocity when $v(t)$ has a positive value.

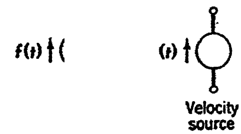


FIG. 2·24. Active elements in translational-mechanical networks.

Mass M is the inertia element. It is the passive element in which kinetic energy of translation is stored. It will be considered to be invariant with time and to be uninfluenced by its motion. If x_M is the displacement of M with respect to a frame of reference, and $v_M \triangleq \dfrac{dx_M}{dt}$ is the velocity of the mass with respect to this frame, the force in the x-direction which will give to M an acceleration of $\dfrac{dv_M(t)}{dt} = \dfrac{d^2x_M(t)}{dt^2}$ is

$$f_M(t) = M\frac{d^2x_M(t)}{dt^2} = M\frac{dv_M(t)}{dt}. \qquad [40]$$

The negative of this force in the x-direction, i.e., $-M\dfrac{d^2x_M(t)}{dt^2}$, is called the reaction force due to inertia. If W is the weight of the body, and g is the acceleration due to gravity, its mass $M = W/g$.

If equation 40 is integrated, it gives for the velocity

$$v_M(t) = M^{-1}\int f_M(t)dt = M^{-1}\int_0^t f_M(t)dt + v_M(0), \qquad [41]$$

in which M^{-1} is called inverse mass.

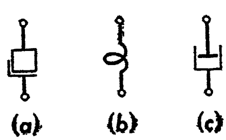


FIG. 2·25. Passive elements in translational-mechanical networks.

The diagrammatic representation for both M and M^{-1} is shown in Fig. 2·25–a. To emphasize the fact that the displacement or velocity of the mass is always taken with respect to some reference, and in order to be able to show in the mechanical-network diagram what this reference is, the rectangle representing the mass has associated with it an L-shaped guide. The element has two terminals, one on the mass itself,

the other on the guide which in turn can be attached to the reference. The advantages of representing M as a two-terminal element will appear later when network diagrams are drawn [FI 1, 2]. To emphasize further the two-terminal nature of the mass, it will be convenient to consider $f_M(t)$ as the force in the x-direction that will produce an acceleration difference of $\dfrac{d_M^2x(t)}{dt}$ " across " M.

Springs provide the restoring forces of the system. If stretched, they try to contract, and thus exert a pull; if compressed, they try to expand, and thus exert a push. For a given stretch or compression, the strength of the pull or push which a spring exerts is dependent upon its translational stiffness K. The spring is the passive element that serves as a reservoir for potential-energy storage. It will be considered invariant with time, and uninfluenced by the amount of the net displacement between its two ends, i.e., the displacement difference " across " it.

The force in the x-direction that will give a displacement difference $x_K(t)$ across a spring of stiffness K is

$$f_K(t) = Kx_K(t) = K\int v_K(t)dt. \qquad [42]$$

If equation 42 is differentiated, the velocity difference across the spring is

$$v_K(t) = K^{-1}\frac{df_K(t)}{dt}, \qquad [43]$$

in which K^{-1} is called translational compliance. The schematic representation of K and K^{-1} as a two-terminal element is shown in Fig. 2·25–b.

If a mechanical system is to be linear its damping forces must be proportional to velocity differences. Damping forces caused by viscous friction and by magnetic induction in coils moving in magnetic fields are in general of this type. As a result of the action of the damping forces, there is a nonreversible transformation of the kinetic energy of the system into heat.

In the network for a translational-mechanical system the passive element that represents an energy sink is translational resistance B. It is considered invariant with time and independent of the velocity difference " across " it.

The force in the x-direction that will give a velocity difference $v_B(t) \triangleq \dfrac{dx_B(t)}{dt}$ across the translational resistance B is

$$f_B(t) = B\frac{dx_B(t)}{dt} = Bv_B(t). \qquad [44]$$

From equation 44, the velocity difference across the translational resistance is

$$v_B(t) = B^{-1}f_B(t), \qquad [45]$$

in which B^{-1} is called translational inverse resistance.

The diagrammatic representation for both B and B^{-1} as a two-terminal element is shown in Fig. 2·25–*c*. It is given the form of a dashpot and is considered to provide viscous friction between piston and cylinder.

Considering now the rotational system, the active elements are the two kinds of sources: torque sources and angular-velocity sources. A known driving torque $\tau(t)$ acting in the system is representable in the system network as a torque source, and a known driving angular velocity $\omega(t)$ of a point in the system is representable in the system network as an angular-velocity source applied at that point. Rotational sources will be represented diagrammatically as shown in Fig. 2·26. The curved arrow on the source indicates the sense of the torque when $\tau(t)$ has a positive value, or the sense of the angular velocity when $\omega(t)$ has a positive value.

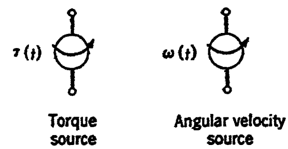


FIG. 2·26. Active elements in rotational-mechanical networks.

For rotation about an axis, the distribution of the mass M with respect to the axis is as important as the mass itself. The inertia element here is the polar moment of inertia $J \triangleq Mr^2$, in which r is the radius of gyration about the axis of rotation. It is the passive element in which kinetic energy of rotation is stored. If θ_J is the angular displacement of J with respect to a frame of reference, and $\omega_J \triangleq \dfrac{d\theta_J}{dt}$ is the angular velocity with respect to this frame, the torque in the θ-direction which will give to J an angular acceleration of $\dfrac{d\omega_J(t)}{dt} = \dfrac{d^2\theta_J(t)}{dt^2}$ is

$$\tau_J(t) = J\frac{d^2\theta_J(t)}{dt^2} = J\frac{d\omega_J(t)}{dt}. \qquad [46]$$

The negative of this torque in the θ-direction, i.e., $-J\dfrac{d^2\theta_J(t)}{dt^2}$ is called the reaction torque due to the moment of inertia.

If equation 46 is integrated, it gives for the angular velocity

$$\omega_J(t) = J^{-1}\int \tau_J(t)dt = J^{-1}\int_0^t \tau_J(t)dt + \omega_J(0), \qquad [47]$$

in which J^{-1} is called inverse moment of inertia.

The diagrammatic representation of both J and J^{-1} as a two-terminal element is the same as for mass, Fig. 2·25–a. In view of this two-terminal nature of the moment of inertia, it will be convenient to consider $\tau_J(t)$ as the torque in the θ-direction that will produce an angular-acceleration difference of $\dfrac{d_J^2\theta(t)}{dt^2}$ "across" J.

In rotation the resilient or springlike bodies exert restoring torques as a result of angular twist, the magnitude of the torque being dependent upon the rotational stiffness K of the body. The torque in the θ-direction that will give an angular-displacement difference $\theta_K(t)$ across a resilient member of rotational stiffness K is

$$\tau_K(t) = K\theta_K(t) = K\int \omega_K(t)dt. \qquad [48]$$

If equation 48 is differentiated, the angular-velocity difference across the resilient member is

$$\omega_K(t) = K^{-1}\frac{d\tau_K(t)}{dt}, \qquad [49]$$

in which K^{-1} is called rotational compliance. The diagrammatic representation of K and K^{-1} as a two-terminal element is the same as in Fig. 2·25–b.

Assuming linear rotational systems in which the damping torques may be caused by viscous friction, the frictional torques will be considered to be proportional to the angular-velocity difference between the surfaces in frictional contact, the constant of proportionality being rotational resistance B. The torque in the θ-direction that will give an angular-velocity difference $\omega_B(t) \triangleq \dfrac{d\theta_B(t)}{dt}$ across the rotational resistance B is

$$\tau_B(t) = B\frac{d\theta_B(t)}{dt} = B\omega_B(t). \qquad [50]$$

From equation 50, the angular-velocity difference across the rotational resistance is

$$\omega_B(t) = B^{-1}\tau_B(t), \qquad [51]$$

in which B^{-1} is called rotational inverse resistance. The diagrammatic representation of B and B^{-1} is shown in Fig. 2·25–c.

It will be noted that the same diagrammatic representations have been given for translational and rotational elements and the same symbols used for their stiffness constants and likewise for their resistance

constants. Where an ambiguity might result as to whether they refer to translational or rotational motion, a subscript r will be given to the rotational constants.

Just as with the electric-network elements, pure elements such as presented above are mathematical fictions. The physical elements possess a combination of the properties ascribed separately to the pure elements. Bodies represented by mass are also resilient, and bodies represented as springs have mass and internal losses effective as damping; and damping devices have masses. It is a question only of the degree of importance of these subordinate properties, and the conditions of a particular case must govern whether they should be included or omitted in the treatment to be made.

18. Newton's Second Law of Motion; D'Alembert's Principle

The differential equations of motion of a mechanical system can be obtained from Lagrange's equations derived by Hamilton's principle from energy conditions in the system, or more directly by use of Newton's second law of motion. Newton's second law, or a slightly different expression of it known as D'Alembert's principle, will be used since it corresponds to the electric-network current law of Kirchhoff.

Newton's second law may be stated as follows:

> If a body is acted upon by several forces, the body is accelerated in the direction of the resultant of these forces, and the magnitude of the acceleration is proportional to this resultant and inversely proportional to the mass of the body.

Assuming the motion is in the x-direction, this can be expressed by

$$M\frac{d^2x(t)}{dt^2} = \sum f(t) \text{ acting in } x\text{-direction.} \qquad [52]$$

Equation 52 can also be written in the form

$$\sum f(t) \text{ acting in } x\text{-direction } -M\frac{d^2x(t)}{dt^2} = 0. \qquad [53]$$

It expresses D'Alembert's principle, namely:

> The sum of the instantaneous external forces acting on a body in a given direction and the body's reaction force in that direction due to inertia is zero.

This principle is convenient to use since it gives a way of converting an equation in statics into an equation in dynamics. The external forces

become functions of time, and a term is added for the reaction due to inertia.

Although Newton's law and D'Alembert's principle have been stated above in the terminology of translatory motion, they apply equally well to rotary motion. Thus for rotation, Newton's law is

$$J\frac{d^2\theta(t)}{dt^2} = \sum\tau(t) \text{ acting in } \theta\text{-direction,} \tag{54}$$

with J, $\theta(t)$, and the resultant torque all taken with respect to the same axis of rotation.

19. Certain Conventions

The positive direction for a force $f(t)$ or a torque $\tau(t)$ will be indicated by an arrow. The force or torque will be considered to be in the arrow direction at any instant at which $f(t)$ or $\tau(t)$ has a positive value.

In a problem concerning rectilinear displacement of a mass with respect to an origin, this origin will be taken at the initial equilibrium position assumed by the mass when the forces supporting it exactly balance the force of gravity, whenever this choice is convenient. The positive direction for a displacement $x(t)$ will be indicated by an arrow, and the displacement will be considered to be in this direction whenever $x(t)$ has a positive value.

Under certain conditions it is convenient to measure the displacement of a moving body with respect to a frame of reference which is moving with constant velocity relative to the original frame of reference. When this is done the selection of the frame controls the magnitude and sign of the displacements.

Likewise, rotational displacements may be taken with respect to either a fixed or an unaccelerated rotating frame. The latter is useful if the body has a steady-state angular velocity, and only the departures from this caused by external disturbing torques are of interest. The positive direction for an angular displacement $\theta(t)$ will be indicated by an arrow, and the displacement will be considered to be in this direction whenever $\theta(t)$ has a positive value.

Application of Newton's law will now be made to a number of simple mechanical systems.

20. One-Coordinate Translational System

In Fig. 2·27 is shown a spring-supported mass M constrained by fixed guides so that it can move only in a vertical direction. A driving force $f(t)$ acts vertically between the frame of reference and M. There is

viscous friction between the mass and its guides which may be represented as a translational resistance B. At $t = 0$, the body is at rest at its equilibrium position, i.e., at a position where the spring is compressed sufficiently to support the weight Mg. The equation of motion of this system will now be formulated.

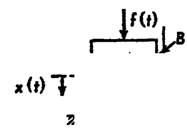


FIG. 2·27. One-coordinate translational system.

The support is assumed to be immovable, and the mass of the spring K is assumed to be negligible. This leaves M the only mass to be considered. Since M can have only a vertical rectilinear motion, a single coordinate is sufficient to describe its position at any instant. Let this coordinate be the downward displacement $x(t)$ of M with respect to its initial equilibrium position.

Using the diagrammatic representations given in Sec. 17 for the mechanical elements, a network diagram can be constructed for the mechanical system as in Fig. 2·28.

Just as in communicating ideas by writing, pictures were used first and then abstract symbols, and in electrical engineering pictorial connection diagrams were used first and then replaced by symbolic diagrams, so now in mechanics pictorial diagrams such as that shown in Fig. 2·27 are being replaced by symbolic diagrams such as that shown in Fig. 2·28. Pictorial diagrams are still useful as an elementary introduction to symbolic diagrams, especially for complicated systems. The procedure [FI 1, 2] for the construction of symbolic diagrams is as follows:

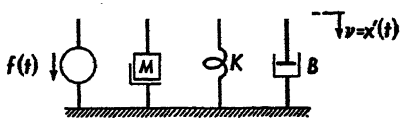


FIG. 2·28. Network diagram of system shown in Fig. 2·27.

a. Identify the two terminals of each active and passive element.
b. Connect together at a common junction those terminals that move together.
c. Connect to the reference junction all terminals that remain stationary with respect to the reference frame.
d. Mark each source element with an arrow to indicate the positive direction of the force or velocity which it represents.
e. Assign a coordinate to each movable junction, and indicate by an arrow the direction for positive values of this coordinate.

The analogy between mechanical networks obtained by this procedure and electric networks will be discussed in Sec. 22.

Referring now to either Fig. 2·27 or Fig. 2·28, and using Newton's second law, the equation of motion is

$$M\frac{d^2x}{dt^2} = f(t) - f_K(t) - f_B(t), \qquad x \triangleq x(t). \tag{55}$$

Or using D'Alembert's principle, it is

$$f(t) - f_K(t) - f_B(t) - M\frac{d^2x}{dt^2} = 0. \tag{56}$$

Substituting the explicit expressions for the forces exerted by the spring and the translational resistance, equation 55 becomes

$$M\frac{d^2x}{dt^2} = f(t) - Kx - B\frac{dx}{dt}. \tag{57}$$

The signs of the terms in equation 57 can be justified as follows: The positive direction of the driving force $f(t)$ is such as to cause a positive acceleration of M. If x is positive the spring is compressed, and pushes upward on M, i.e., tends to give M a negative acceleration. If $\frac{dx}{dt}$ is positive the mass is moving downward against frictional forces which push upward on M, i.e., tend to give M a negative acceleration.

Equation 57 can be written

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t). \tag{58}$$

As a result of choosing the reference for x at the initial equilibrium position of M, the initial conditions are

$$x(0) = 0, \quad \text{and} \quad x'(0) = 0.$$

21. Two-Coordinate Translational System

The mass M_1 of Fig. 2·29 is supported by a spring upon mass M_2. The latter is supported elastically upon a frame that moves vertically with a known displacement $x_3(t)$ with respect to a fixed reference, downward being chosen positive. Both masses are constrained by guides to move only in a vertical direction; there is viscous friction between the masses and the guides. Designate the equivalent translational resistances by B_1 and B_2 and the stiffness constants of the springs by K_1 and K_2. The masses have initial displacements from their initial equilibrium positions and initial velocities as follows:

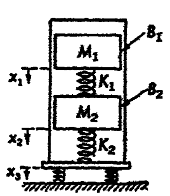


FIG. 2·29. Two-coordinate translational system.

MASS	DISPLACEMENT	VELOCITY
M_1	a_1 units downward	b_1 units upward
M_2	a_2 units upward	b_2 units downward

The formulation of the differential equations for the system will now be carried out.

The two masses, as well as the frame, can move independently, but only in a vertical direction. Consequently the dependent variables are taken as the displacements downward of these two masses from their initial equilibrium positions. The displacement downward of the frame is known.

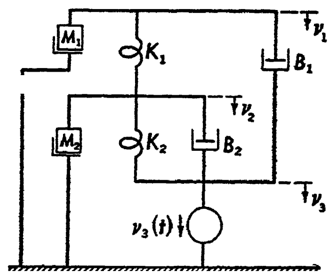


FIG. 2·30. Network diagram of system shown in Fig. 2·29.

The network diagram for the system is given in Fig. 2·30. Since the velocity of the frame is known, a velocity source $\nu_3(t) \triangleq x_3'(t)$ is used as the driving source of the network.

Newton's second law, applied first to M_1 and then to M_2, gives the two equations

$$M_1 \frac{d^2x_1}{dt^2} = K_1(x_2 - x_1) + B_1 \frac{d}{dt}(x_3 - x_1),$$
$$M_2 \frac{d^2x_2}{dt^2} = -K_1(x_2 - x_1) + K_2(x_3 - x_2) + B_2 \frac{d}{dt}(x_3 - x_2), \quad [59]$$

in which $x_1 \triangleq x_1(t)$, $x_2 \triangleq x_2(t)$, and $x_3 \triangleq x_3(t)$. The signs of the terms in the right members of equations 59 can be justified by the following reasoning: If x_2 is greater than x_1, spring K_1 is stretched and exerts a pull downward on M_1, and this tends to give M_1 a positive acceleration. On the other hand this stretching of K_1 exerts an upward pull on M_2 and tends to give M_2 a negative acceleration. If $\frac{dx_3}{dt}$ is greater than $\frac{dx_1}{dt}$, the guide is moving downward faster than M_1. This exerts a pull downward on M_1 and tends to give it a positive acceleration. Similarly the pull on M_2 is downward if $\frac{dx_3}{dt}$ is greater than $\frac{dx_2}{dt}$.

Equations 59 can be put in the symmetric form

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1x_1 - K_1x_2 = B_1 \frac{dx_3}{dt} \triangleq f_1(t),$$

$$-K_1x_1 + M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + (K_1 + K_2)x_2 = B_2 \frac{dx_3}{dt} + K_2x_3 \triangleq f_2(t). \qquad [60]$$

Since $x_3(t)$ and its derivative $v_3(t)$ are known functions of time, $f_1(t) \triangleq B_1v_3(t)$ is a known force applied to M_1; likewise,

$$f_2(t) \triangleq B_2v_3(t) + K_2 \int v_3(t)dt = B_2v_3(t) + K_2 \int_0^t v_3(t)dt + K_2x_3(0)$$

is a known force applied to M_2. In terms of the mechanical network (Fig. 2·30), a velocity source $v_3(t)$ in series with B_1 has been replaced by a force source $f_1(t)$ in parallel with B_1. A velocity source in series with a parallel group composed of B_2 and K_2 has been replaced by two force sources that add to give a single force source $f_2(t)$ in parallel with B_2 and K_2. The new mechanical network is given in Fig. 2·31.

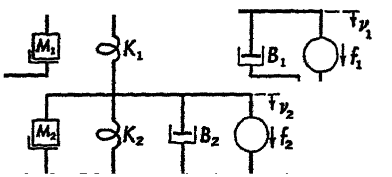


FIG. 2·31. Network diagram of Fig. 2·30 after substitution of two force sources for a velocity source.

This substitution of force sources for velocity sources is similar to the exchange of sources (Sec. 14) that can be made in electric networks. A summary of certain useful source exchanges that can be made in mechanical networks and in electric networks and still maintain terminal conditions invariant is given in Table 2.

Since the origins for the coordinates were chosen at the initial equilibrium positions of the masses, the initial conditions for the system of Fig. 2·31 are

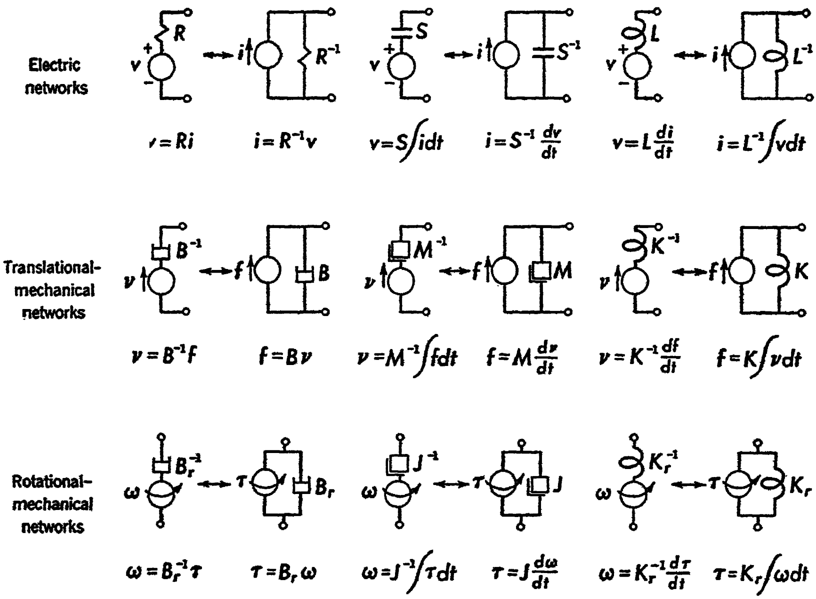
$$x_1(0) = a_1, \qquad x_1'(0) = -b_1,$$

$$x_2(0) = -a_2, \qquad x_2'(0) = b_2.$$

22. FORMULATION OF ELECTRIC ANALOGS

Analogous systems were defined in Sec. 15 as different physical representations of the same set of i-d equations. The electric analog for a

TABLE 2. EXCHANGES OF SOURCES WHICH MAINTAIN TERMINAL CONDITIONS INVARIANT



mechanical system can be formed by finding an electric network having a set of i-d equations of the same form as the set for the mechanical system. In many cases the analog can be arrived at intuitively, but a more reliable way to obtain a correct analog is to (1) write the equations for the prototype system, (2) rewrite the equations using electric-network constants and variables, and (3) interpret these network equations by sketching the network whose behavior they describe.

It has been noted in Sec. 15 that, within certain limitations of realizability, electric networks have duals. There is thus a possibility of there being more than one electric analog of a mechanical system. As an illustration of this, consider a problem in translation. The linear constant-coefficient equations of mechanical systems derived by application of Newton's second law are equations of forces. For the electric system the equations of the same type derived by application of Kirchhoff's laws are equations of either currents or voltages. It is possible then to have $i \sim f$ (read "current analogous to force") [HA 2] or to have $v \sim f$. The two networks thus derived will be duals since node-pair voltages in one will be analogous to loop currents in the other.

Consider, for example, what the analogous electric networks are for the mechanical system shown in Fig. 2·27. Equation 58 for this system is rewritten here in terms of velocity $\nu \triangleq \dfrac{dx}{dt}$ to facilitate comparison. It is

$$M\frac{d\nu}{dt} + B\nu + K\int \nu dt = f(t). \tag{58}$$

Making $i \sim f$, the analogous equation of currents is

$$C\frac{dv}{dt} + Gv + \Gamma\int v dt = i(t). \tag{61}$$

This is the equation for the one-node-pair network of Fig. 2·32. This electric network has the same geometric pattern or topological form (Sec. 2) as the mechanical network of Fig. 2·28 and could have been

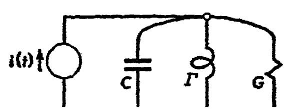


FIG. 2·32. Electric analog of mechanical system shown in Fig. 2·27. Based on $i \sim f$.

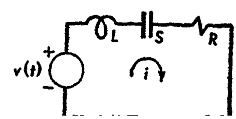


FIG. 2·33. Electric analog of mechanical system shown in Fig. 2·27. Based on $v \sim f$.

derived directly on the basis of this topological identity by making the substitutions:

$$\begin{array}{rlll} & \text{current source } i(t) & \text{for} & \text{force source } f(t) \\ \text{passive elements} & \left\{\begin{array}{l} C \\ G \\ \Gamma \end{array}\right. & \begin{array}{l}\text{for}\\ \text{for}\\ \text{for}\end{array} & \begin{array}{l} M \\ B \\ K \end{array} \\ & \text{unknown } v(t) & \text{for} & \nu(t). \end{array}$$

Making $v \sim f$, the equation of voltage drops that is analogous to equation 58 is

$$L\frac{di}{dt} + Ri + S\int i dt = v(t), \tag{62}$$

which is the equation for the one-loop network of Fig. 2·33. Topologically, this electric network is the dual (Sec. 15) of the mechanical network of Fig. 2·28 and of the electric network of Fig. 2·32. It can be derived directly on the basis of this topological duality by (1) sketching

the topological dual of the mechanical network and (2) inserting in this topological dual

$$\begin{array}{rlll} \text{voltage source} & v(t) & \text{for} & \text{velocity source } \nu(t) \\ \text{passive elements} & \begin{cases} L \\ R \\ S \end{cases} & \begin{matrix} \text{for} \\ \text{for} \\ \text{for} \end{matrix} & \begin{matrix} K^{-1} \\ B^{-1} \\ M^{-1} \end{matrix} \\ \text{unknown} & i(t) & \text{for} & f(t). \end{array}$$

If equation 61 is differentiated once there is obtained

$$C\frac{d^2v}{dt^2} + G\frac{dv}{dt} + \Gamma v = \frac{di}{dt}. \qquad [63]$$

This equation in node-current derivatives has essentially the same form as equation 58 when the latter is written in terms of displacements rather than velocities. The network that corresponds to equation 63 differs from that shown in Fig. 2.32 only by having the derivative of the source current prescribed rather than the current. It represents an analogous network in which $\frac{di}{dt} \sim f$, $i \sim \int f dt$, and $v \sim x$.

Three analogous networks have been developed for the mechanical system of Fig. 2·27. Others can be developed similarly by iteration of the process of differentiation or of integration, starting with equations 61 and 62. A summary of the analogous constants, variables, and geometric forms of network diagrams of mechanical and electric systems for four bases of analogy is given in Table 3. Other bases can be established by shearing the column of electric variables with respect to the column of mechanical variables, as for example, by making $\frac{d^2v}{dt^2} \sim f$. Note that, as in the transition from equation 61 to equation 63 by a differentiation, this does not change the correspondence of electric and mechanical constants, or change the network connections.

The principles that have been applied here in deriving electric analogs of a simple mechanical network are equally applicable to more complicated mechanical networks [Ol 1], except that a dual is not obtainable if the original network is nonplanar. These principles, together with those governing exchange of sources, make it possible to establish analogs both geometrically and analytically with little difficulty.

A tabular comparison of the loop and node bases of analysis of translational-mechanical networks is given in Table 4. Part of the material presented in Division 3 of this table will be discussed below in Sec. 27; certain other parts of the table will be dealt with in problems.

TABLE 3. ANALOGOUS CONSTANTS, VARIABLES, AND GEOMETRIC FORMS OF NETWORK DIAGRAMS OF MECHANICAL AND ELECTRIC SYSTEMS

	Mechanical Systems		Electric Systems			
	Translation	Rotation	$i \sim f$ or τ	$\frac{di}{dt} \sim f$ or τ	$v \sim f$ or τ	$\frac{dv}{dt} \sim f$ or τ
Constants	M B K	J B_r K_r	C G Γ	C G Γ	L R S	L R S
Variables	⋮	⋮	⋮	⋮	⋮	⋮
	$\int f dt$	$\int \tau dt$	$\int i dt$, q	i	$\int v dt$	v
	f	τ	i	$\frac{di}{dt}$	v	$\frac{dv}{dt}$
	$\frac{df}{dt}$	$\frac{d\tau}{dt}$	$\frac{di}{dt}$	$\frac{d^2i}{dt^2}$	$\frac{dv}{dt}$	$\frac{d^2v}{dt^2}$
	⋮	⋮	⋮	⋮	⋮	⋮
	⋮	⋮	⋮	⋮	⋮	⋮
	x	θ	$\int v dt$	v	$\int i dt$, q	i
	$\frac{dx}{dt}$, v	$\frac{d\theta}{dt}$, ω	v	$\frac{dv}{dt}$	i	$\frac{di}{dt}$
	$\frac{d^2x}{dt^2}$, a	$\frac{d^2\theta}{dt^2}$, α	$\frac{dv}{dt}$	$\frac{d^2v}{dt^2}$	$\frac{di}{dt}$	$\frac{d^2i}{dt^2}$
	⋮	⋮	⋮	⋮	⋮	⋮
Geometric Form of Network Diagram	Diagram is based on connections of the mechanical system.		Same form as for the mechanical system.		Dual (if realizable) of the form for the mechanical system.	

The similarity of Table 4 to Table 1 for electric networks makes unnecessary a detailed explanation here of Table 4.

23. ONE-COORDINATE ROTATIONAL SYSTEM

A torsional pendulum consisting of a shaft having fixed ends and a flywheel at its midpoint is shown in Fig. 2·34. The two lengths of shaft

each have a rotational stiffness $K/2$. The flywheel has a moment of inertia J. The pendulum's torsional oscillations are damped by viscous friction, the rotational resistance being denoted by B. The flywheel is rotated through an initial angle ϕ_1 in a counterclockwise sense, viewed from the right end of the shaft, and then released. The equation of motion will be written.

The network diagram for the system is shown in Fig. 2·35.

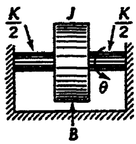


FIG. 2·34. One-coordinate rotational system.

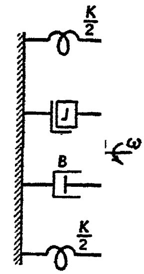


FIG. 2·35. Network diagram for system of Fig. 2·34.

Let the angular twist of the shaft from its equilibrium position viewed from the right be $\theta(t)$. Then the equation of motion for the flywheel is

$$J\frac{d^2\theta}{dt^2} = -K\theta - B\frac{d\theta}{dt}, \qquad \theta \triangleq \theta(t). \tag{64}$$

The signs of the terms in the right member may be explained as follows: A positive angular displacement θ twists the shaft in a counterclockwise direction. The restoring torque is thus directed clockwise, and tends to give J a negative acceleration. A positive angular velocity $\dfrac{d\theta}{dt}$ causes a frictional torque directed clockwise and this also tends to give J a negative acceleration.

Since the initial angle of twist ϕ_1 is in a counterclockwise sense, the initial conditions are

$$\theta(0) = \phi_1, \qquad \theta'(0) = 0.$$

24. Two-Coordinate Rotational System

A shaft bearing two rotors (Fig. 2·36) is set in torsional oscillation by a fluctuating driving torque $[\tau_c + \tau(t)]$ applied to rotor 1, τ_c being the constant part of the torque. This constant torque is balanced by a constant load torque and constant frictional torques, the sum of all of these being represented by τ_c applied to rotor 2. The moments of inertia of the rotors are J_1 and J_2; the rotational stiffness of the shaft connecting them is K. The fluctuation of each rotor about its constant angular velocity is damped by viscous friction, the rotational resist-

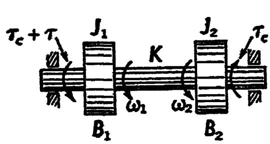
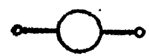
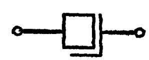
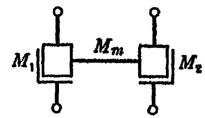


FIG. 2·36. Two-coordinate rotational system.

TABLE 4. TR. ATIONAL-MECHANICAL NETWOR
COMPARISON OF P AND NODE BASES OF ANAL

	Loop Basis		Diagrammatic Symbol		
1	$\rightarrow$ Indicates direction of velocity when $\nu(t)$ has a positive value.	$\vec{\nu}(t)$ Velocity	 Source	Force	$\rightarrow$ dicates dire force when f positive valu
	$\nu_B = B^{-1} f_B$	B^{-1} Inverse translational resistance	 Sink (heat)	ional ice	$_B = B\nu_B$
2	$\nu_K = K^{-1} \dfrac{df_K}{dt}$	K^{-1} Translational compliance	 Reservoir (potential)	K Translationa stiffness	$\int \nu_K dt$
	$\nu_M = M^{-1} \int f_M dt$	M^{-1} Inverse mass	 Reservoir (kinetic) Self-mass (only)		$M \dfrac{d\nu_M}{dt}$

3	$\tilde{M}_1^{-1} \triangleq \dfrac{M_2}{M_1M_2 - M_m^2} = \dfrac{1}{M}\dfrac{l_1^2 + h^2}{h^2}$ $\tilde{M}_2^{-1} \triangleq \dfrac{M_1}{M_1M_2 - M_m^2} = \dfrac{1}{M}\dfrac{l_2^2 + h^2}{h^2}$ $\tilde{M}_m^{-1} \triangleq \dfrac{M_m}{M_1M_2 - M_m^2} = \dfrac{1}{M}\dfrac{l_1l_2 - h^2}{h^2}$	$\tilde{M}^{-1}$ Inverse mass for end 1 (2 free) $\tilde{M}_2^{-1}$ Inverse mass for end 2 (1 free) $\tilde{M}_m^{-1}$ Inverse mutual mass	 Reservoir (kinetic) Self-mass and mutual mass	M_1 Self-mass for end 1 (2 fixed) M_2 Self-mass for end 2 (1 fixed) M_m Mutual mass	$M_1 \triangleq M\dfrac{l_2^2 + h^2}{l^2}$ $M_2 \triangleq M\dfrac{l_1^2 + h^2}{l^2}$ $M_m \triangleq \dfrac{l_1l_2 - h^2}{l^2}$
4	—	$l = e - n + s$ Number of independent geometric loops.	—	$n_p = n - s$ Number of independent geometric node-pairs.	A stationary or an unaccelerated node is taken as reference.
5	Number of loop equations needed.	$u_f = l - f_{\text{sources}}$ Number of unknown loop forces.	—	$u_v = n_p - v_{\text{sources}}$ Number of unknown node-pair velocities.	Number of node equations needed.
6	Velocity rise is taken as positive.	Around a closed path $\sum_{k=1}^{l} v_k(t) = 0$	—	At a common node $\sum_{k=1}^{n_p} f_k(t) = 0$ D'Alembert's principle	Force in positive x-direction for the node is taken as positive.

Explanation of Divisions of Table:

1. Active elements.
2. Passive elements.
3. Mutual mass; network representation of a bar (see Sec. 27 for meaning of symbols).
4. Independent loops and node-pairs in geometric pattern of network.
5. Number of dependent variables needed.
6. Law of mechanics used in writing equations.

ances being B_1 and B_2. The initial angular positions and angular velocities of the rotors are those resulting from the constant torque τ_c. The equations of motion for the system will be formulated.

Here the constant angular velocity ω_c which is produced by the constant torque τ_c is not of interest; consequently only the variations from this caused by $\tau(t)$ need to be considered in the equations of motion. A rotating but unaccelerated frame of reference will therefore be used.

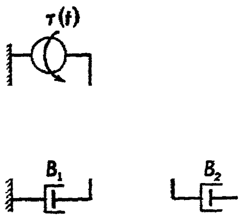


FIG. 2·37. Network diagram for system of Fig. 2·36. References are rotating at constant angular velocity ω_c.

Viewed from the right end, the positive direction for the driving torque $\tau(t)$ is counterclockwise. Let ω_1 and ω_2 be the *changes* in the angular velocities of rotors 1 and 2, respectively, from ω_c of the system, taking counterclockwise increases as positive, as indicated in Fig. 2·36.

The network diagram for the system is given in Fig. 2·37.

Application of Newton's second law first to J_1 and then to J_2 gives the two equations

$$\left.\begin{aligned} J_1\frac{d\omega_1}{dt} &= \tau(t) - K\int(\omega_1 - \omega_2)dt - B_1\omega_1, \\ J_2\frac{d\omega_2}{dt} &= K\int(\omega_1 - \omega_2)dt - B_2\omega_2, \end{aligned}\right| \qquad [65]$$

in which $\omega_1 \triangleq \omega_1(t)$ and $\omega_2 \triangleq \omega_2(t)$. If the integral $\int(\omega_1 - \omega_2)dt$ is positive, rotor 1 has gained on rotor 2 and the left end of the shaft is twisted counterclockwise with respect to the right end. The restoring torque developed by the shaft is then such as to give J_1 a negative acceleration and J_2 a positive acceleration. When ω_1 is positive, the damping torque is directed clockwise and tends to give J_1 a negative acceleration. It is similar with ω_2 and J_2. Equations 65 can be written in the symmetric form

$$\left.\begin{aligned} J_1\frac{d\omega_1}{dt} + B_1\omega_1 + K\int\omega_1 dt - K\int\omega_2 dt &= \tau(t), \\ -K\int\omega_1 dt + J_2\frac{d\omega_2}{dt} + B_2\omega_2 + K\int\omega_2 dt &= 0. \end{aligned}\right| \qquad [66]$$

Note that by inspection of the mechanical-network diagram (Fig. 2·37) and the use of the node basis of analysis equations 66 could have been written directly, which amounts in effect to an application of D'Alembert's principle to each rotor separately.

Turning now to the statement of the initial conditions and recalling that displacements and velocities were measured from the equilibrium values corresponding to τ_c, the initial conditions to be used are

$$\omega_1(0) = 0, \qquad \omega_1^{(-1)}(0) - \omega_2^{(-1)}(0) = 0.$$
$$\omega_2(0) = 0,$$

25. Two-Coordinate Rotational System with Coupling through Moment of Inertia

Three short rotating shafts (Fig. 2·38–a) are connected through a differential gear (Fig. 2·38–b) which makes shaft 3 have an angular velocity r times the difference between the angular velocities of shafts 1 and 2. The shafts with their gears have moments of inertia J_a, J_b, and J_c. The external driving torques applied to the shafts are $\tau_1(t)$, $\tau_2(t)$, and $\tau_3(t)$. The equations of motion for the system will be formulated.

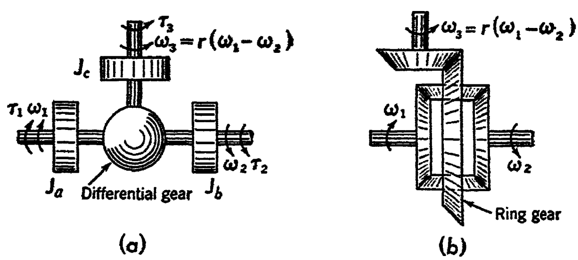


FIG. 2·38. Three shafts connected by a differential gear. Details of the differential gear are shown in b.

Let ω_1, ω_2, and ω_3 be the angular velocities of the shafts, their positive directions being indicated by arrows in Fig. 2·38–a. For convenience in writing the equations let τ_d be the equivalent load torque exerted upon shaft 1 by the differential gear. Conversely the reaction of τ_d will be a driving torque exerted through the differential gear upon shafts 2 and 3. It will be noted from Fig. 2·38–b that the gear ratios between the shafts are such as to make $\omega_1/\omega_2 = 1$ when $\omega_3 = 0$; $\omega_3/\omega_1 = r$ when $\omega_2 = 0$; and $\omega_3/\omega_2 = -r$ when $\omega_1 = 0$. These are derivable also from the relation

$$\omega_3 = r(\omega_1 - \omega_2). \tag{67}$$

Applying Newton's second law to shafts 1, 2, and 3 in turn, the following equations can be written:

$$J_a \frac{d\omega_1}{dt} = \tau_1 - \tau_d, \tag{68}$$

$$J_b \frac{d\omega_2}{dt} = \tau_2 + \tau_d, \tag{69}$$

$$J_c \frac{d\omega_3}{dt} = \tau_3 + \frac{\tau_d}{r}. \tag{70}$$

Eliminating τ_d from equations 68 and 70,

$$J_a \frac{d\omega_1}{dt} + rJ_c \frac{d\omega_3}{dt} = \tau_1 + r\tau_3. \tag{71}$$

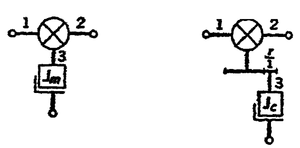
Eliminating τ_d from equations 69 and 70,

$$J_b \frac{d\omega_2}{dt} - rJ_c \frac{d\omega_3}{dt} = \tau_2 - r\tau_3. \tag{72}$$

Finally, eliminating ω_3 by use of relation 67, equations 71 and 72 become

$$\begin{aligned} (J_a + r^2J_c) \frac{d\omega_1}{dt} - r^2J_c \frac{d\omega_2}{dt} &= \tau_1 + r\tau_3, \\ -r^2J_c \frac{d\omega_1}{dt} + (J_b + r^2J_c) \frac{d\omega_2}{dt} &= \tau_2 - r\tau_3. \end{aligned} \tag{73}$$

Equations 73 can be interpreted as the equations of a two-coordinate rotational system with coupling through moment of inertia [PA 4]. In other words, the differential gear affords a way of realizing mutual moment of inertia in rotational-mechanical systems.



(a) (b)

FIG. 2·39. Graphical symbols for representing mutual moment-of-inertia in rotational-network diagrams.

If mutual moment of inertia is represented by J_m, then for this example $J_m = r^2J_c$. In rotational-network diagrams J_m can be represented by the three-terminal symbol shown in Fig. 2·39–a. An optional form displaying J_c and the gear ratio $r/1$ is shown in Fig. 2·39–b.

The definition of mutual moment of inertia can be derived from equations 73. If τ_3 is zero, so that shaft 3 is left free to turn, and sufficient torque τ_1 is applied to shaft 1 to keep it from accelerating when

shaft 2 is given an angular acceleration $\frac{d\omega_2}{dt}$, then

$$J_m = -\frac{\tau_1}{\frac{d\omega_2}{dt}}.$$

Thus J_m is numerically equal to the torque which must be applied to shaft 1 to keep it from accelerating when shaft 2 is given a unit angular acceleration and shaft 3 is free to turn.

Similarly, J_m is numerically equal to the torque which must be applied to shaft 2 to keep it from accelerating when shaft 1 is given unit angular acceleration and shaft 3 is free to turn.

The network diagram for the system of Fig. 2·38 is shown in Fig. 2·40. The electric analog with $i \sim \tau$ is given in Fig. 2·41–*a*, and the analog with $v \sim \tau$ is given in Fig. 2·41–*b*.

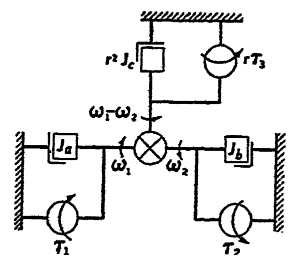


FIG. 2·40. Network diagram for system shown in Fig. 2·38.

In the above development τ_1, τ_2, and τ_3 have been considered external applied torques so that the mechanical-network equivalent of the differential gear could be established. More generally, τ_1, τ_2, and τ_3 would be produced by combinations of active and passive network elements.

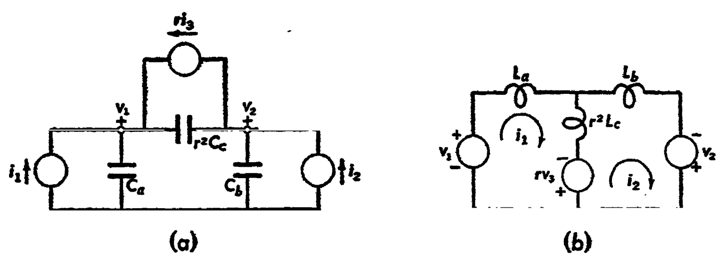


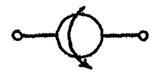
FIG. 2·41. Electric analogs of mechanical system shown in Fig. 2·38: *a* with $i \sim \tau$; *b* with $v \sim \tau$.

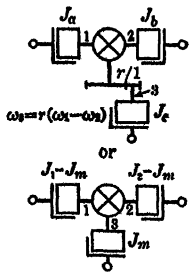
A summary comparison of the loop and node bases of analysis of rotational-mechanical networks is given in Table 5. This table is arranged identically with Table 4 for translational-mechanical networks and Table 1 for electric networks in order to emphasize the network analogy between the mechanical and electric systems.

Various exchanges of sources in rotational-mechanical networks which maintain terminal conditions invariant are included in Table 2.

TAB ROTATIONAL-MECHANICAL NE ORKS

MPA LOOP AND NODE BASES OF NALYS

	Loop Basis		Diagrammatic Symbol	Node Basis	
1	Indicates direction of angular velocity when $\omega(t)$ has a positive value.	$\omega(t)$ Angular velocity	 Source	$\tau(t)$ Torque	Indicates direction of torque when $\tau(t)$ has a positive value.
	$\omega_B = B_r^{-1}\tau_B$	B_r^{-1} Inverse rotational-resistance.	 Sink (heat)	B_r Rotational resistance	$\tau_B = B_r\omega_B$
2	$\omega_K = K_r^{-1}\dfrac{d\tau_K}{dt}$	K_r^{-1} Rotational compliance	 Reservoir (potential)	K_r Rotational stiffness	$\tau_K = K_r\int \omega_K dt$
	$\omega_J = J^{-1}\int \tau_J dt$	J^{-1} Inverse moment-of-inertia.	 Reservoir (kinetic) Self-mi (only)	J Moment of inertia (abbreviation mi)	$\tau_J = J\dfrac{d\omega_J}{dt}$

3	$\mathcal{J}_1^{-1} \triangleq \dfrac{J_2}{J_1J_2 - J_m^2}$	$\mathcal{J}_1^{-1}$ Inverse mi of shaft 1 (2 free)		J_1 Self-mi of shaft 1 (2 fixed)	$J_1 \triangleq J_a + J_m$
	$\mathcal{J}_2^{-1} \triangleq \dfrac{J_1}{J_1J_2 - J_m^2}$	$\mathcal{J}_2^{-1}$ Inverse mi of shaft 2 (1 free)		J_2 Self-mi of shaft 2 (1 fixed)	$J_2 \triangleq J_b + J_m$
	$\mathcal{J}_m^{-1} \triangleq \dfrac{J_m}{J_1J_2 - J_m^2}$	$\mathcal{J}_m^{-1}$ Inverse mutual mi	Reservoir (kinetic) Self-mi and mutual mi	J_m Mutual mi	$J_m \triangleq r^2J_c$
4	—	$l = e - n + s$ Number of independent geometric loops.	—	$n_p = n - s$ Number of independent geometric node-pairs.	A stationary or an unaccelerated node is taken as reference.
5	Number of loop equations needed.	$u_\tau = l - \tau_{\text{sources}}$ Number of unknown loop torques.	—	$u_\omega = n_p - \omega_{\text{sources}}$ Number of unknown node-pair angular velocities.	Number of node equations needed.
6	Angular velocity rise is taken as positive.	Around a closed path $\sum_{k=1}^{l} \omega_k(t) = 0$	—	At a common node $\sum_{k=1}^{n_p} \tau_k(t) = 0$ D'Alembert's principle	Torque in positive θ-direction for the node is taken as positive.

Explanation of Divisions of Table:

1. Active elements.
2. Passive elements.
3. Mutual moment-of-inertia; network representation of a differential gear.
4. Independent loops and node-pairs in geometric pattern of network.
5. Number of dependent variables needed.
6. Law of mechanics used in writing equations.

26. One-Coordinate System with Combined Translation and Rotation

So far, only translation alone and rotation alone have been considered. In this section, and in the one following, a simple example will be given of a system in which there is combined translation and rotation.

In Fig. 2·42–*a* are shown the essential features of a seismic instrument for measuring rectilinear vibrations by means of a rotational-mechanical system. It consists of a frame supporting a pendulum which can swing only in a vertical plane. The frame is clamped to the body whose vibration is to be measured and has the same vertical motion as that body.

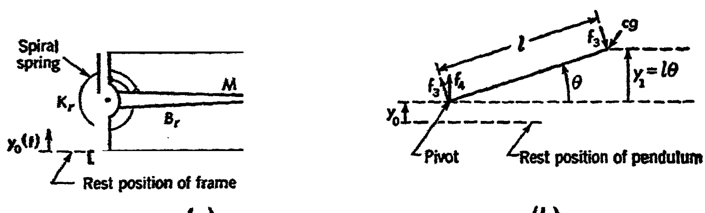


FIG. 2·42. One-coordinate system combining translation and rotation.

The pendulum has a mass M. Its center of gravity (cg) lies at a distance l from the pivot, and its radius of gyration about a horizontal line through the cg is h. A spiral spring of rotational stiffness K_r tends to keep the pendulum in a horizontal position. The angular motion of the pendulum with respect to the frame is damped, the rotational resistance being B_r.

Let the frame start from rest at $t = 0$ and have a vertical displacement thereafter of $y_0(t)$, upward being positive. The pendulum is at rest initially with respect to the frame. The equation for the subsequent rectilinear displacement of the cg of the pendulum relative to the frame will be written, assuming that the angular swing of the pendulum is small.

The pendulum is shown in a general displaced position in Fig. 2·42–*b*. The displacement $y_0(t)$ of the pivot is measured from a stationary reference. $\theta(t)$ is the angular displacement of the axis of the pendulum relative to the frame. $y_1(t)$ is the displacement of the cg relative to the frame, and since the angular displacement is small, $y_1(t) = l \sin \theta(t) \approx l\theta(t)$.

When the pendulum has an angular displacement θ and an angular velocity $\frac{d\theta}{dt}$ the change in the torque exerted by the frame on the pendu-

lum is $K_r\theta + B_r \frac{d\theta}{dt}$. This may be replaced by an equivalent torque lf_3 produced by forces f_3 acting through the *cg* and the pivot. If in addition the frame is accelerated vertically there is an additional force f_4 acting on the pendulum at the pivot.

The motion of the pendulum is planar and can be considered to be a combination of (1) translation with every point moving the same as the *cg* and (2) rotation about a line through the *cg* and normal to the plane of motion. In other words, the motions of translation and rotation can be treated separately.

Considering first translation without rotation, the absolute displacement of the *cg* is $(y_0 + y_1)$, and

$$M\frac{d^2}{dt^2}(y_0 + y_1) = f_3 + f_4 - f_3 = f_4. \qquad [74]$$

Since the angles are small it is permissible to replace the vertical components of forces f_3 by the total forces. Considering next rotation about a line through the *cg* without translation, the moment of inertia about this line is Mh^2, and

$$Mh^2\frac{d^2\theta}{dt^2} = -lf_3 - lf_4. \qquad [75]$$

Substituting for lf_3 and f_4,

$$Mh^2\frac{d^2\theta}{dt^2} = -\left(K_r\theta + B_r\frac{d\theta}{dt}\right) - lM\frac{d^2}{dt^2}(y_0 + y_1). \qquad [76]$$

Replacing θ by y_1/l and $M(h^2 + l^2)$ by J, there is obtained

$$\frac{J}{l^2}\frac{d^2y_1}{dt^2} + \frac{B_r}{l^2}\frac{dy_1}{dt} + \frac{K_r}{l^2}y_1 = -M\frac{d^2y_0}{dt^2}. \qquad [77]$$

In equation 77, $J \triangleq M(h^2 + l^2)$ is the moment of inertia of the pendulum about a transverse horizontal line through the pivot. It is composed of the moment of inertia Mh^2 about a parallel line through the *cg* plus the product of the mass M and the square of the distance l between the lines. This is in fact the parallel-axis theorem for finding the moment of inertia about a line that does not pass through the *cg*.

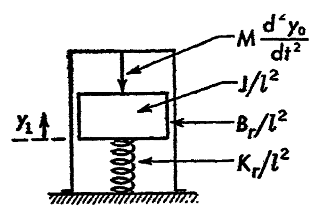


FIG. 2-43. Translational system equivalent to the pendulum system shown in Fig. 2-42.

Equation 77 is an equation for rectilinear motion. The division of the rotational constants J, B_r, and K_r

by l^2 has converted them to translational constants. The equivalent translational system for the pendulum system is shown in Fig. 2·43 and is similar to the translational system treated in Sec. 20 except for the positive direction for the displacement.

Since the pendulum was at rest relative to the frame at $t = 0$, the initial conditions are $y_1(0)$ and $y_1'(0) = 0$.

27. TWO-COORDINATE SYSTEM WITH COMBINED TRANSLATION AND ROTATION

The rigid bar of Fig. 2·44–*a* is supported on springs at its ends and constrained so that it can vibrate only in a vertical plane [Ti 1]. The radius of gyration of the bar about a horizontal axis through the bar's center of gravity is h. The mass of the bar is M. The springs, the stiffness constants of which are K_1 and K_2, support the bar in a horizontal

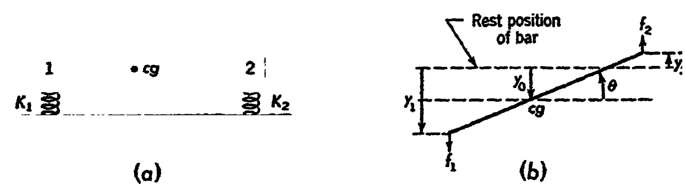


FIG. 2·44. Two-coordinate system combining translation and rotation.

position when at rest. The damping of the system can be considered negligible. The bar is depressed until each end has an initial displacement of a_1 units downward and is then suddenly released. The equations of motion for the system will be written and the initial conditions expressed.

When the bar is released, its motion will be a combination of bouncing and pitching. Its departure from its equilibrium position can be expressed in terms of a translation downward $y_0(t)$ of the center of gravity (*cg*) plus a rotation counterclockwise $\theta(t)$ about an axis through the *cg* and normal to the plane of rotation (Fig. 2·44–*b*). Thus the two coordinates, y_0 and θ, are used to describe the position of any specified point of the bar at any instant. These two coordinates become the dependent variables of the system.

Disregarding the springs for the present, let $f_1(t)$ and $f_2(t)$ represent two external forces acting at the ends of the bar, the positive direction for f_1 being taken downward and the positive direction for f_2 being taken upward.

Considering first translation and then rotation, the application of Newton's second law gives the two equations of motion

$$\left.\begin{aligned} M\frac{d^2y_0}{dt^2} &= f_1 - f_2, \\ Mh^2\frac{d^2\theta}{dt^2} &= l_1f_1 + l_2f_2, \end{aligned}\right\} \tag{78}$$

in which Mh^2 is the moment of inertia of the bar about the transverse horizontal axis passing through the *cg*.

Equations 78 can be made symmetrical by a change of variables. Let the displacements of the ends of the bar from their equilibrium positions be $y_1(t)$ and $y_2(t)$ as indicated in Fig. 2·44–*b*. Then

$$\left.\begin{aligned} y_1 &= y_0 + l_1\theta, \\ y_2 &= -y_0 + l_2\theta. \end{aligned}\right\} \tag{79}$$

Differentiating equations 79 twice with respect to t and solving for the second derivatives of y_0 and θ, there is obtained

$$\left.\begin{aligned} \frac{d^2\theta}{dt^2} &= \frac{1}{l}\left(\frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2}\right), \\ \frac{d^2y_0}{dt^2} &= \frac{1}{l}\left(l_2\frac{d^2y_1}{dt^2} - l_1\frac{d^2y_2}{dt^2}\right), \end{aligned}\right\} \tag{80}$$

in which $l \triangleq l_1 + l_2$. Substitution of expressions 80 in equations 78 yields

$$\left.\begin{aligned} M\frac{l_2}{l}\frac{d^2y_1}{dt^2} - M\frac{l_1}{l}\frac{d^2y_2}{dt^2} &= f_1 - f_2, \\ M\frac{h^2}{l}\frac{d^2y_1}{dt^2} + M\frac{h^2}{l}\frac{d^2y_2}{dt^2} &= l_1f_1 + l_2f_2. \end{aligned}\right\} \tag{81}$$

Solution of equations 81 for f_1 and f_2 gives the symmetric equations

$$\begin{aligned} M\frac{l_2^2 + h^2}{l^2}\frac{d^2y_1}{dt^2} - M\frac{l_1l_2 - h^2}{l^2}\frac{d^2y_2}{dt^2} &= f_1, \\ -M\frac{l_1l_2 - h^2}{l^2}\frac{d^2y_1}{dt^2} + M\frac{l_1^2 + h^2}{l^2}\frac{d^2y_2}{dt^2} &= f_2. \end{aligned} \tag{82}$$

The equations of the spring-supported bar can be made complete by substituting explicit expressions for the forces. These expressions are

$$\left.\begin{aligned} f_1 &= -K_1 y_1, \\ f_2 &= -K_2 y_2. \end{aligned}\right\} \qquad [83]$$

These signs may be substantiated as follows: If y_1 is positive, the spring K_1 is compressed and pushes upward on the left end of the bar, making f_1 negative for this condition. If y_2 is positive, the spring K_2 is elongated and pulls downward on the right end of the bar, making f_2 negative for this condition.

The initial conditions are

$$\begin{aligned} y_1(0) &= a_1, & y_1'(0) &= 0, \\ y_2(0) &= -a_1, & y_2'(0) &= 0. \end{aligned}$$

Equations 82 describe a two-coordinate translational system with coupling through mass, the bar providing a way of realizing mutual mass in translational-mechanical systems. Referring to ends 1 and 2 of Fig. 2·44–a, and using the parallel-axis theorem to identify J_1 and J_2 about axes through ends 2 and 1, respectively,

$$M_1 \triangleq M\frac{l_2^2 + h^2}{l^2} = \frac{J_1}{l^2}, \quad \text{self-mass for 1, with 2 fixed,}$$

$$M_2 \triangleq M\frac{l_1^2 + h^2}{l^2} = \frac{J_2}{l^2}, \quad \text{self-mass for 2, with 1 fixed,}$$

$$M_m \triangleq M\frac{l_1 l_2 - h^2}{l^2}, \quad \text{mutual mass.}$$

If v_1 and v_2 are the velocities of ends 1 and 2, respectively, then equations 82 can be written

$$\left.\begin{aligned} M_1\frac{dv_1}{dt} - M_m\frac{dv_2}{dt} &= f_1, \\ -M_m\frac{dv_1}{dt} + M_2\frac{dv_2}{dt} &= f_2. \end{aligned}\right. \qquad [84]$$

A definition of mutual mass may be obtained from equations 84. It is

$$M_m \triangleq \begin{cases} -\dfrac{f_1}{\dfrac{dv_2}{dt}}, & \text{with } \dfrac{dv_1}{dt} \text{ zero, or} \\[2ex] \dfrac{f_2}{\dfrac{dv_1}{dt}}, & \text{with } \dfrac{dv_2}{dt} \text{ zero.} \end{cases}$$

Stated in words, mutual mass M_m is numerically equal to the force necessary to hold end 1 from being accelerated when the acceleration of end 2 is unity. It is also numerically equal to the force necessary to hold end 2 from being accelerated when the acceleration of end 1 is unity.

Mutual mass can also be identified in certain expressions for a physical pendulum. If a pendulum is struck a transverse blow at a point other than the center of percussion, a transverse force is exerted by the pivot on the pendulum. Consider the bar to be a pendulum pivoted at end 1 and subjected to a transverse blow at end 2 producing an acceleration $\frac{d^2y_2}{dt^2}$ of end 2. The force of the transverse blow is then $M_2\frac{d^2y_2}{dt^2}$, and the transverse force exerted by the pivot is

$$\left(1 - \frac{l_1 l}{l_1^2 + h^2}\right) M_2 \frac{d^2y_2}{dt^2} \equiv -M_m \frac{d^2y_2}{dt^2}.$$

This follows since

$$\left(1 - \frac{l_1 l}{l_1^2 + h^2}\right) M_2 = \left(1 - \frac{l_1 l}{l_1^2 + h^2}\right) \frac{l_1^2 + h^2}{l^2} M$$

$$\frac{l_1 l_2 - h^2}{l^2} M \triangleq -M_m$$

A diagrammatic representation of mutual mass is included in Fig. 2-45. The relations in this network are displayed in equations 84.

A summary of the relations for the bar is given in division 3 of Table 4.

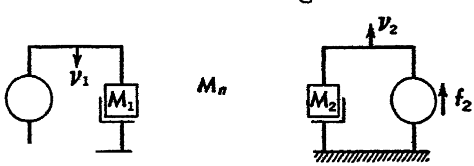


FIG. 2-45. Network diagram of a bar with external impressed forces. It introduces a symbol for mutual mass.

28. Electric Analogs of the Rigid Bar

The rigid bar, and the lever which is a pivoted bar, are very common mechanical coupling devices. They have their analogs among the electric-network coupling devices. The analog on the $v \sim f$ basis is found by first rewriting equations 84 as equations of voltages. They become

$$L_1 \frac{di_1}{dt} - L_m \frac{di_2}{dt} = v_1,$$

$$-L_m \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = v_2, \qquad [85]$$

in which L_m has been used for mutual inductance to avoid confusion with the use of M for mass. These are the equations of a transformer without losses, the network being as in Fig. 2·46–a or Fig. 2·46–b. The analogous elements can be determined by comparison of the coefficients in equations 85 and 84.

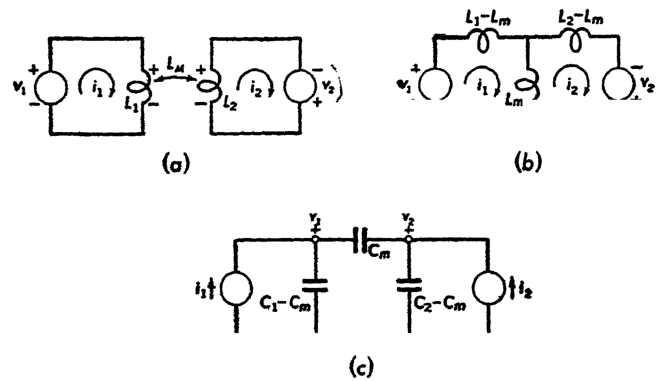


FIG. 2·46. Electric analogs of the bar with external impressed forces. a and b are based on $v \sim f$; c is based on $i \sim f$.

Two coupled circuits such as shown in a or b of Fig. 2·46 have a coefficient of coupling $k \triangleq L_m/\sqrt{L_1L_2}$. Analogous to this, the bar of Fig. 2·44–a has a coefficient of coupling. It can be found by substitution of the mechanical equivalents of L_m, L_1, and L_2 in the definition of k. This gives

$$k = \frac{l_1 l_2 - h^2}{\sqrt{(l_1^2 + h^2)(l_2^2 + h^2)}}. \qquad [86]$$

It is seen that $k = 0$ when $h^2 = l_1 l_2$, and $k = 1$ when $h = 0$. When $k = 0$ the bar will pivot about either point 1 or point 2, i.e., a motion of point 1 will cause no motion at point 2, and conversely a motion of point 2 will cause no motion at point 1. The condition $k = 1$ can be approximated by concentrating the mass at the center of gravity. The system tends then to pivot about its cg. A second way in which k can be made unity is to pivot the bar at its cg to some fixed support. The bar then becomes a simple force-multiplying mechanism. If the friction in the pivots is negligible, then the analog is an ideal transformer having unity coupling, no exciting current, and no losses.

The analog on the $i \sim f$ basis is found by rewriting equations 84 as

equations of currents. They become

$$\left.\begin{aligned} C_1 \frac{dv_1}{dt} - C_m \frac{dv_2}{dt} &= i_1, \\ -C_m \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} &= i_2, \end{aligned}\right\} \qquad [87]$$

in which the node-pair voltages v_1 and v_2 are the dependent variables analogous to velocities ν_1 and ν_2. The electric network represented by equations 87 is shown in Fig. 2·46–c. Comparison of coefficients in equations 87 and 84 discloses the analogous elements. The network of Fig. 2·46–c is topologically the same as the mechanical network of Fig. 2·45 and is the topological dual of the network of Fig. 2·46–b.

C. LUMPED-CONSTANT ELECTROMECHANICAL SYSTEMS

Having shown the procedure for treating electric systems and mechanical systems, there remains to be shown the procedure for treating them when they occur in combination. Here the careful specification of variables, choice of reference systems for the variables, and choice of units is especially important.

29. ONE-LOOP ELECTRIC SYSTEM AND ONE-COORDINATE MECHANICAL SYSTEM COMBINED

In Fig. 2·47 is shown an elementary electromechanical system for the conversion of electric energy into the energy of rectilinear mechanical motion such as is used in the electrodynamic loud speaker. The mass M is spring supported and frictionally damped,

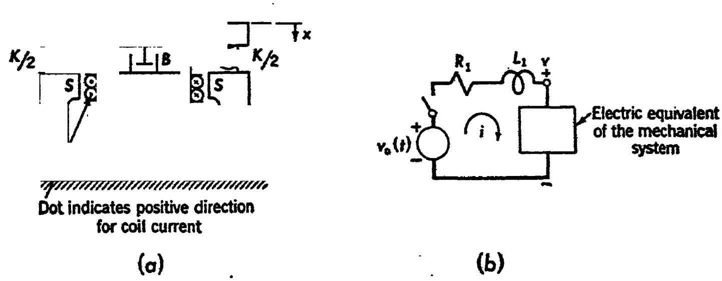


FIG. 2·47. One-loop, one-coordinate electromechanical system.

the spring stiffness being K and the mechanical resistance being B. Attached to the mass is a current-carrying coil of N turns lying in a radial magnetic field of flux density β provided by the cylinder-shaped magnet with concentric poles. The coil is cylindrical, the radius of its mean turn

being r. The coil has a resistance R_1 and self-inductance L_1. At $t = 0$ the switch is closed, applying the driving voltage $v_0(t)$ to the coil. The mass at that instant is at rest at its equilibrium position. The equations of this electromechanical system will now be formulated.

The system is composed of a one-coordinate mechanical system and a one-loop electric system. Let $x(t)$ be the downward displacement of M from its equilibrium position, and let $i(t)$ be the coil current. The choice of the arrow direction for i in the loop fixes the positive direction for i in the coil winding. Assume it is as indicated by the dots and crosses on the cross sections of the conductors in Fig. 2·47–*a*.

By Kirchhoff's voltage law, the equation for the electric circuit is

$$L_1 \frac{di}{dt} + R_1 i + U \frac{dx}{dt} = v_0(t). \tag{88}$$

Here $U \triangleq 2\pi r N\beta$. It is called the *electromechanical coupling constant*. $U \dfrac{dx}{dt}$ is the emf induced in the coil by its motion normal to the magnetic field. When $\dfrac{dx}{dt}$ is positive, the coil is moving downward through the field. The polarity of the motional emf is such as to make it a drop in potential in the arrow direction.[1] The sign of the term is thus $+$.

By Newton's second law of motion, the equation for the mechanical system is

$$M \frac{d^2x}{dt^2} = -Kx - B \frac{dx}{dt} + Ui. \tag{89}$$

U is the same electromechanical coupling constant appearing in equation 88. It is assumed that the units used for every constant and variable appearing in the two equations are taken from the same system, e.g., the mks (meter-kilogram-second) practical system, or the cgs (centimeter-gram-second) absolute electromagnetic system, or any other single system of units [HA 3]. This eliminates from the equations any dimensionless factors such as powers, or inverse powers, of 10.

The coil conductors, carrying current i, lie normal to the magnetic field and are acted upon by a force Ui. When i is positive, the coil is

[1] This may be established by use of Fleming's right-hand rule for the direction of a motional emf. An alternative rule, easier to remember and usually more convenient to apply, is as follows: Extend the fingers of the right hand in the positive direction for the flux density in which the conductor moves, i.e., from N to S, with palm in such a position that the motion is toward the palm. The thumb, if pointed along the conductor, will then point in the direction of the motional emf.

urged downward.[1] This downward force tends to give M a positive acceleration, so the sign of the term is +.

For x positive, the spring is compressed and pushes upward on M, tending to give M a negative acceleration, so the sign of the Kx term is negative. If $\frac{dx}{dt}$ is positive, the motion of M is downward. The frictional force is directed upward, tending to give M a negative acceleration, so the sign of the $B\frac{dx}{dt}$ term is negative.

The initial conditions for the system can be expressed as follows:

$$i(0) = 0,$$
$$x(0) = 0, \qquad x'(0) = 0.$$

30. ALL-ELECTRIC ANALOG FOR AN ELECTROMECHANICAL SYSTEM; ALL-MECHANICAL ANALOG

The mechanical portion of the electromechanical system treated in Sec. 29 can be replaced by an electric-network equivalent. Since this equivalent must function as part of the original electric circuit, and the coil must have the same coil current as with the combined electric and mechanical systems, the magnitudes of the constants in the equivalent network must be chosen accordingly.

As a first step in the procedure, equations 88 and 89 are rewritten in the form

$$L_1\frac{di}{dt} + R_1 i = v_0(t) - U\frac{dx}{dt}, \qquad [90]$$

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = Ui. \qquad [91]$$

Then the change of variable $v \triangleq U\frac{dx}{dt}$ is made. Equation 91 is also divided by U. There results

$$L_1\frac{di}{dt} + R_1 i = v_0(t) - v, \qquad [92]$$

$$\frac{M}{U^2}\frac{dv}{dt} + \frac{B}{U^2}v + \frac{K}{U^2}\int v\,dt = i. \qquad [93]$$

[1] This may be established by use of the right-hand rule for the force. An alternative rule is as follows: Grasp the current-carrying conductor by the right hand with the thumb pointing in the direction of positive current and the fingers extended in the positive direction for the flux density in which the conductor moves. The force on the conductor will be directed away from the palm of the hand.

From equation 93 it is seen that within a single system of units the factor converting mechanical constants (M,B,K) into electrical constants (C,G,Γ) is U^{-2}. Designating the equivalent electrical constants as follows:

$$C = \frac{M}{U^2}, \qquad G = \frac{B}{U^2}, \quad \text{and} \quad \Gamma = \frac{K}{U^2},$$

the all-electric network equations are

$$L_1 \frac{di}{dt} + R_1 i = v_0(t) - v, \tag{94}$$

$$C\frac{dv}{dt} + Gv + \Gamma \int v dt = i. \tag{95}$$

The all-electric network based on equations 94 and 95 is shown in Fig. 2·48–*a*. It is the all-electric analog of the electromechanical system on the $i \sim f$ basis. If the voltage source in this network is replaced by a current source (see Table 2), the network is that shown in Fig. 2·48–*b*.

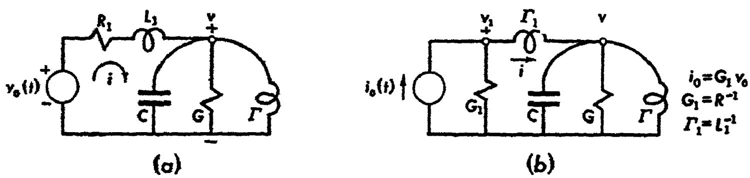


FIG. 2·48. All-electric analogs of the electromechanical system of Fig. 2·47. Based on $i \sim f$.

If an all-mechanical analog of the electromechanical system is desired, it can be derived either analytically from equations 92 and 93 or graphically from the network diagram in Fig. 2·48–*b*. With $f \sim i$ the network diagram of the all-mechanical analog is that shown in Fig. 2·49.

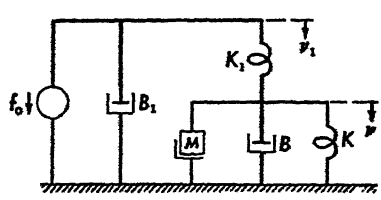


FIG. 2·49. All-mechanical analog of the electromechanical system of Fig. 2·47. Based on $f \sim i$.

The relation between analogous electrical and mechanical constants can be established by recalling that (1) the mechanical system produces a voltage drop $v = U\nu$ in the electric circuit and (2) the electric system produces a force $f = Ui$ in the mechanical circuit. For example, the relation between R and B can be found from the electrical equation $i = Gv$ by substituting for i and v, obtaining $U^{-1}f = GU\nu$. This gives the mechanical equation $f = GU^2\nu = B\nu$, in which $B = U^2G$. Further-

more, in any single system of units, the dimensions of U^2G are those of B. Similarly it can be shown that $K = U^2\Gamma$ and $M = U^2C$.

31. SUMMARY

In this chapter it has been shown how the variables can be chosen and their reference systems established in one-dimensional electric and mechanical systems. The i-d equations have been formulated for electric networks on both the loop basis and the node basis. The differential equations have been formulated for mechanical systems having translational motion, rotational motion, and a combination of these motions. The principles by which the electric analogs of these systems can be formed have been shown. Finally, the differential equations for a combined electric and mechanical system have been formulated and certain all-electric and all-mechanical analogs developed.

PROBLEMS

2·1. The diagram shows a bridged-T filter section with terminal resistances. $R^2 = L/C$.

(a) Give the number of independent geometric loops and number of independent geometric node-pairs.

(b) Write the i-d equations for the network, using the loop basis.

(c) Write the i-d equations using the node basis.

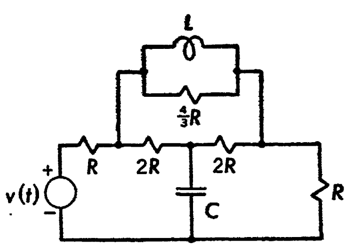


FIG. 2·P1

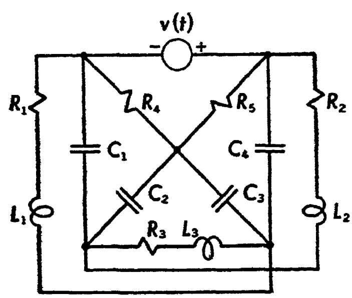


FIG. 2·P2

2·2. For the network shown in the diagram give the following information: (a) Number of independent geometric loops, (b) number of independent geometric node-pairs, (c) the i-d equations for the network, using the loop basis.

2·3. In the network illustrated, switch K is closed at $t = 0$. The initial energy storage in the condensers and inductances is zero.

(a) Give the number of independent geometric loops and number of independent geometric node-pairs.

(b) Give the number of dependent variables necessary if the i-d equations are written on the loop basis. Give the number if the node basis is used.

(c) If the voltage source is replaced with a current source, what is the relation between these sources?

(d) Write the i-d equations for the network using the node basis.

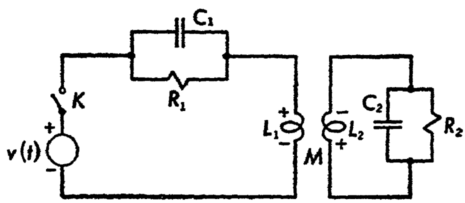


FIG. 2·P3

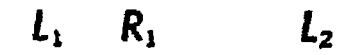


FIG. 2·P4

2·4. The network illustrated is in the steady state when the relay K opens. V is a constant direct voltage. Letting the branch currents be as indicated, give the initial values ($t = 0+$) of:

(a) i_1, i_1', and i_1''.

(b) i_2, i_2', and i_2''.

2·5. The network illustrated is in the steady state with the switch open. At $t = 0$ this switch is closed. (a) Give the initial value and indicate the initial polarity or direction of v_{S_1}, v_{S_2}, and i_L. (b) Write the i-d equations for the network with K closed. (c) Give the initial value ($t = 0+$) of i_{R_1}, i_{R_2}, i_L', and v_{S_2}'.

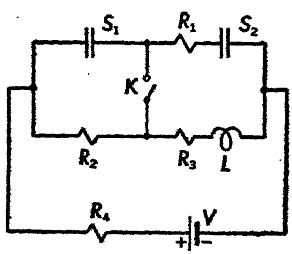


FIG. 2·P5

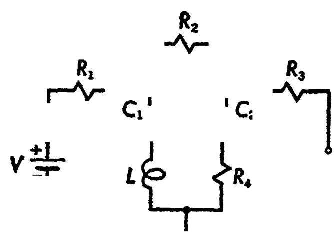


FIG. 2·P6

2·6. In the network shown in the diagram the switch is closed until steady-state conditions are reached and then it is opened. (a) What is the initial value of the voltage across the switch after it opens? (b) What is the initial value of the first derivative of this voltage?

2·7. The network illustrated is in the steady state when switch K closes at $t = 0$. Write the i-d equations for the network and express the initial conditions.

2·8. The diagram shows a three-phase rectifier with inductive load L_2 and "wave smoothing" inductances L_1. There is equal mutual inductance M between each two inductances L_1. The phase order of the balanced sinusoidal voltage drops at the terminals abc is v_{ab}, v_{bc}, v_{ca}. Neglect the voltage drop in the arc. The interval of overlap during which the arc is transferred from one anode to the next to conduct

may be neglected, i.e., each anode may be assumed to conduct for one third of the cycle.

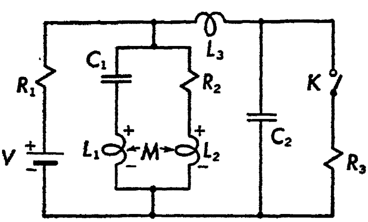


FIG. 2·P7

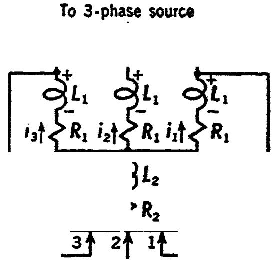


FIG. 2·P8

(*a*) Using the branch currents indicated, write the differential equations holding for the third of the cycle in which anode 1 is conducting.

(*b*) For boundary conditions, express the relations that hold among the initial and final values of these branch currents in this interval.

2·9. The network illustrated is the equivalent network of a two-stage vacuum-tube amplifier with conductance-capacitance coupling G_2C_4, and a conductance load G_4. The tubes are triodes and the amplification constant of each tube is μ.

(*a*) Write the i-d equations for the network using the basis which is more favorable. Assume v_0 is a direct voltage of 1 volt.

(*b*) If the condensers have no charge at $t = 0-$, find $v_3(0+)$. For brevity, let C_{11}, C_{22}, and C_{33} denote the self-capacitances of nodes 1, 2, and 3, respectively.

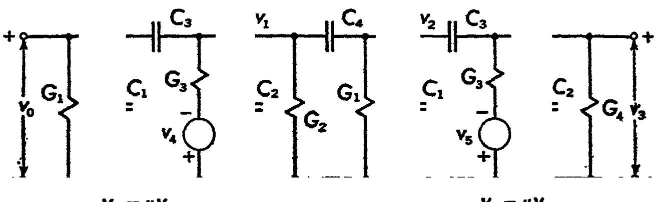


FIG. 2·P9

2·10. With the ignition system shown in the steady state, contact K opens at $t = 0$. (*a*) Write for the system the i-d equations that hold for the interval between

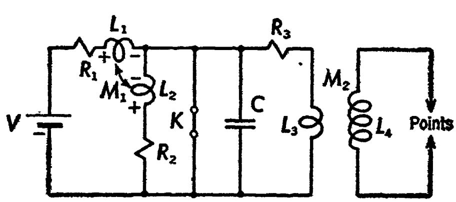


FIG. 2·P10

the opening of K and the sparkover between the points. (*b*) Give the necessary initial conditions.

2·11. A tripping network used in the experimental study of transients is shown in the diagram. When the voltage which builds up slowly across gap G reaches γ volts this gap breaks down and serves as a short-circuiting switch. The voltage across R_2 is used to trip the oscillograph, and the voltage across C_4 is used to start the transient in the circuit which is under observation.

(a) Express the initial conditions in this tripping network.

(b) Write the differential equations for this network.

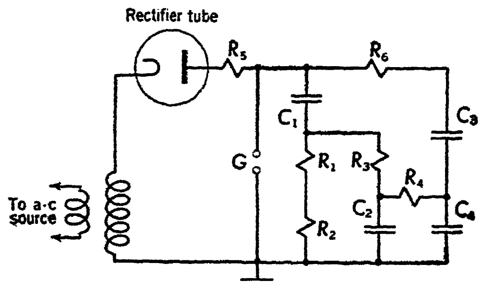


FIG. 2·P11

2·12. (a) Write the set of i-d equations for the general network having n independent node-pairs. (b) Express this set of equations by means of a typical term of a typical equation using index notation (see equation 24).

2·13. A simple nonplanar network is shown in the diagram. Demonstrate that it is impossible to form a physically realizable dual for a network having this geometric form.

$R_1 = 1$ ohm	$R_4 = 4$ ohms	$R_7 = 7$ ohms
$R_2 = 2$ ohms	$R_5 = 5$ ohms	$R_8 = 8$ ohms
$R_3 = 3$ ohms	$R_6 = 6$ ohms	$R_9 = 9$ ohms

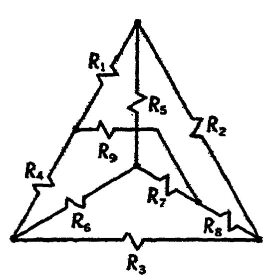


FIG. 2·P13

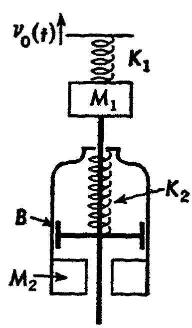


FIG. 2·P14

2·14. The mass M_1 is suspended from a support by the spring of stiffness constant K_1. Attached to M_1 is a vibration damper consisting of a dashpot, spring, and mass. The plunger which carries the piston is rigidly attached to M_1. The chamber of mass M_2 is attached to and supported by the spring K_2. The lower end of K_2 is attached to the piston. The viscous friction of the dashpot can be represented by the trans-

lational resistance B. Assume that the upper end of spring K_1 is given an up-and-down sinusoidal motion.

(*a*) Write the differential equations for the system.

(*b*) Sketch the mechanical-network diagram.

(*c*) Write the differential equations for the analogous electric network with $v \sim f$, and sketch this network.

(*d*) Repeat part (*c*) using $i \sim f$.

2·15. In the mechanical system illustrated, the mass M_1 is given translatory horizontal motion through the spring K_1, one end of which has the known forced velocity $v_0(t)$. Attached to M_1 is a vibration damper consisting of dashpot, spring K_2, and mass M_2. The viscous friction within the dashpot, between the dashpot and the supporting surface, and between M_1 and the supporting surface can be represented, respectively, by the translational resistances B_2, B_3, and B_1. Assume that the masses are at their equilibrium positions at $t = 0$.

(*a*) Sketch the mechanical-network diagram and write the i-d equations for the system.

(*b*) Sketch the analogous electric network having $i \sim f$, and write the i-d equations for this network.

(*c*) Sketch the analogous electric network having $v \sim f$, and write the i-d equations for this network.

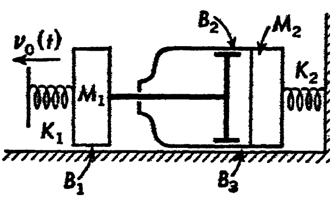


FIG. 2·P15

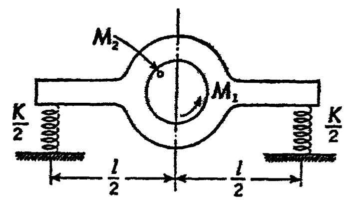


FIG. 2·P16

2·16. A motor is mounted on a carriage supported at its ends on springs. The carriage is constrained so that it can vibrate only up and down in the vertical plane. The rotor assembly is slightly unbalanced by attaching a small mass M_2 at a distance r from the axis of the shaft. With the rotor revolving at a constant angular velocity above the critical angular velocity for the system and the entire system vibrating in the steady state, the power to the motor is cut off at the instant when the carriage is at the lowest position in its travel. The system is to be considered linear.

Write the differential equation that describes the instantaneous equilibrium conditions in the system during the decrease of rotor angular velocity, and give the initial conditions.

The mass of the carriage and motor before unbalancing is M_1. The translational resistance for vibrations of the carriage is B. Assume that the angular velocity of the rotor after the power is cut off is $\omega_0 e^{-at}$.

2·17. Three short rotating shafts are connected through a differential gear that makes shaft 3 have an angular velocity r times the difference between the angular velocities of shafts 1 and 2. The shafts with their gears and flywheels have moments of inertia J_1, J_2, and J_3. Shafts 1 and 2 are subject to viscous damping, the damping torque on each being proportional to the angular velocity of the shaft with

respect to a stationary casing. Shaft 3 is connected through a hydraulic clutch with a driving shaft. The torque transmitted by the clutch is B_3 times the angular velocity difference between the two elements of the clutch. The system, including the driving shaft, is at rest at $t = 0$. The angular velocity thereafter of the driving shaft is $\omega_4(t) = a(1 - e^{-bt})$.

(a) Write the differential equations and sketch the mechanical-network diagram for the system.

(b) Show the analogous electric network based on $i \sim \tau$.

(c) Repeat part (b) using the $v \sim \tau$ basis.

2·18. Assume that an automobile can be represented by the much-simplified mechanical system illustrated. The radius of gyration of the frame about its *cg* is h. Neglect sidewise roll of the car and consider only pitching and bouncing through small displacements. Consider the frame to be rigid.

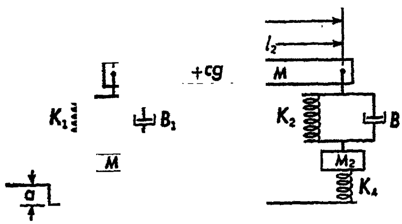


FIG. 2·P18

(a) Write the differential equations of motion for the system. The road surface has the form given. The car is moving to the left with velocity v.

(b) Write the differential equations for the analogous electric network with $i \sim f$ and sketch this network.

(c) Repeat part (b) for the analogous electric network with $v \sim f$.

2·19. A long rotor of total mass M rotating with constant angular velocity ω is supported on pedestal bearings which permit vibratory motion of the shaft in the horizontal plane. The effective translational stiffness and translational resistance for motion in this plane are as indicated in the diagram.

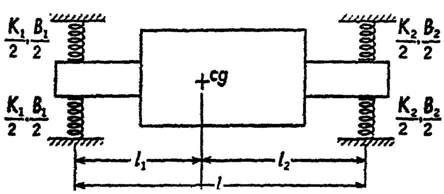


FIG. 2·P19

Assume that the rotor is statically but not dynamically balanced. The dynamical unbalance can be represented by two small equal masses m lying in the same axial plane but on opposite sides of the axis of rotation. These masses are equal distances a from the axis of rotation; their planes of rotation are equal distances $b/2$ from the center of gravity. Assume that the angle of vibration of the shaft in the horizontal plane remains small. Assume that the radius of gyration of the rotor about a vertical axis through the *cg* is h.

(*a*) Write the differential equations of motion of the system, and sketch the mechanical-network diagram.

(*b*) Sketch the analogous electric network with $i \sim f$.

(*c*) Sketch the analogous electric network with $v \sim f$.

2·20. (*a*) Show that if a rigid bar of mass M is to be treated mathematically as an equivalent rigid massless bar bearing three concentrated masses, one at the center of gravity and the other two at distances l_1 and l_2, respectively, either side of the center of gravity, these masses should be as follows:

$$M_{cg} = M\frac{l_1 l_2 - h^2}{l_1 l_2}$$

$$M_{l_1} = M\frac{h^2}{l_1^2 + l_1 l_2}$$

$$M_{l_2} = M\frac{h^2}{l_2^2 + l_1 l_2}$$

Here h is the radius of gyration about an axis through the center of gravity.

(*b*) Show the analogy to a transformer with leakage inductances.

2·21. In the mechanical system illustrated, two unequal masses M_1 and M_2 are connected by a cable-and-pulley system in such a way that their translational motions with respect to the frame are equal and opposite. There is viscous friction between M_1 and M_2, and between M_1 and the frame. A spring attached to M_2 and the frame tends to keep the masses in a central position. The mass of the cable and of the pulleys may be neglected. The frame is attached to a vibrating body that gives it translational motion represented by x_3 with respect to a reference.

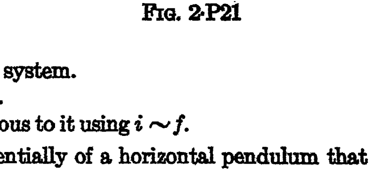


FIG. 2·P21

(*a*) Write the differential equations for the system.

(*b*) Show the mechanical-network diagram.

(*c*) Show the electric network that is analogous to it using $i \sim f$.

2·22. One form of seismograph consists essentially of a horizontal pendulum that swings like a gate. The pendulum consists of a horizontal beam pivoted at one end in a conical socket and supported at the other by a stay wire running back diagonally to a point of attachment considerably above the pivot. The beam bears a large mass near its free end. The pivot is attached to the earth and moves with the earth's crust when there is a crustal disturbance while the pendulum mass tends at the outset to remain in its original position. There is viscous damping that provides a damping torque proportional to the angular velocity of the pendulum. The relative motion between the earth and the pendulum's center of gravity (*cg*) is recorded graphically by a suitable electric linkage consisting of a magnet, electric circuit, and galvanometer.

With all the system initially at rest, the earth surface experiences a sudden crustal disturbance expressible as a horizontal *velocity* $y'(t) \triangleq Y_m e^{-pt} \sin \omega_1 t$, $0 \leqq t$, in a direction normal to the pendulum. The displacement $y(t)$ of the earth's crust is measured from its undisturbed position. All displacements are small and the system may be treated as linear.

(*a*) Write the differential equations for this electromechanical system and express the initial conditions. (*b*) Find an all-electric analog.

The constants in a single system of units are as follows: For the pendulum, M is the total mass, K is the rotational stiffness, B is the rotational resistance, l is the distance between pivot and *cg*, and h is the radius of gyration of the beam about a vertical axis through the *cg*. For the electric circuit, R is the total resistance and L is the total self-inductance. For the galvanometer, J_1 is the moment of inertia, K_1 is the torsional stiffness, and B_1 is the torsional resistance. The electromechanical coupling constants are U_1 for pendulum and electric circuit and U_2 for electric circuit and galvanometer.

2·23. An electromagnetic pick-up for investigating mechanical vibrations is shown in the diagram. The motion of the armature of mass M relative to the core changes the length of the air gap in the central leg and the reluctance of the magnetic circuit. Starting with the system initially at rest, the core is given a vertical velocity $Ae^{-pt}\sin\omega_1 t$.

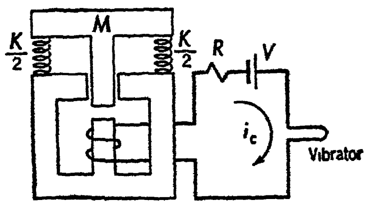


FIG. 2·P23

Write the differential equations for the system, considering that all units are taken from a single system.

The length of the air gap is $x_0 - x$, the current in the coil is $i_0 + i$, and the electrodynamic force on the armature is $f_0 + f$. Here x_0, i_0, and f_0 are the initial equilibrium values and x, i, and f are increments. For x small, assume that the self-inductance $L(x) \triangleq L_0 + ax$. The voltage drop across the self-inductance is $\frac{d}{dt}[L(x)i_c]$ and the force on the armature is $f_0 + f \triangleq \frac{1}{2}i_c^2 \frac{dL(x)}{dx}$. For x small, the square of i and the derivative of the product xi may be neglected. The effect of the vibrator on the circuit is negligible.

2·24. An electromechanical system using a varying capacitance for coupling is shown in the diagram. With the system initially at rest an external force $F_m \sin\omega_1 t$ is applied to the movable plate at $t = 0$.

Write the differential equations of the system, considering that all units are taken from a single system.

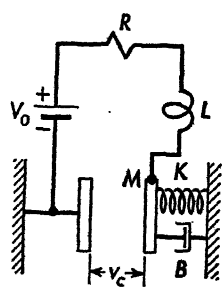


FIG. 2·P24

The area of the movable plate is A and the dielectric present is air. The distance between the plates is $x_0 - x$, the charge on the plates is $q_0 + q$, and the force on the movable plate due to the charges is $f_0 + f$. Here x_0, q_0, and f_0 are the initial equilibrium values and x, q, and f are increments. The voltage drop v_c across the varying capacitance is $(q_0 + q)/C(x)$ and the force on the movable plate is $f_0 + f \triangleq \frac{1}{2}v_c^2 \frac{dC(x)}{dx}$. For x small, the square of q and the product xq may be neglected. The elongation of the spring produced by q_0 is h_0.

CHAPTER III

AN INTRODUCTION TO THE $\mathfrak{L}$ TRANSFORMATION AND ITS INVERSE

In the preceding chapter attention was directed to the formulation of one-dimensional integrodifferential equations of the type whose solution is simplified by the $\mathfrak{L}$-transformation method. In the present chapter the nature of the $\mathfrak{L}$ transformation will be brought out so that it, together with its inverse, may be applied intelligently in solving equations of this type as well as others to be treated later.

1. METHOD OF LOGARITHMS ANALOGOUS TO METHOD OF $\mathfrak{L}$ TRANSFORMATION

The best-known and simplest analogous method is the method of logarithms in arithmetic. This well-known method can be discussed in the language of *transformations,* as will be shown by treating the arithmetic problem of multiplying two numbers.

One way of finding the product of two numbers is to use a table of logarithms. In its simplest form such a table (Table 1) consists of two columns of numbers which are associated in pairs. In the column labeled *Original* appear the original numbers. In the column labeled *Transform* appear the numbers obtained from the original numbers by a transformation. This particular transformation consists of an expression of the original number as a power of 10. This power (exponent, or logarithm) is the transform. The transformation is called finding the logarithm.

TABLE 1. NUMBER TRANSFORMS

Original	Transform
1	0
10	1
100	2

If the columns are arranged as in Table 1, the result of the logarithmic transformation is obtained by entering the table in the left-hand column and passing from the original to its transform in the right-hand column. If the table is used in the opposite direction the transformation is called

the *inverse* logarithmic transformation, and the first transformation is then called the *direct* transformation.

The familiar solution of a multiplication problem by logarithms can be described using this terminology. Applying the direct transformation, that is, proceeding from left to right in a more complete table of the type of Table 1, the transforms of the original factors are found. Next these transforms are added to find an intermediate result. The inverse transformation is then applied to the intermediate result, i.e., the table is used from right to left. The number corresponding to the intermediate result is the final result or product.

This familiar use of the logarithmic transformation serves to simplify the multiplication problem. The simplification results, because, by use of the transformation table, the operation of multiplication is replaced by the simpler operation of addition. Similarly, the operation of addition might be replaced by that of multiplication by using the table from right to left and then left to right, but this is not done because it makes the addition problem harder rather than easier.

Used in the ordinary way the logarithmic transformation simplifies the solution of arithmetic problems because it replaces the operations of multiplication, division, involution, and evolution by the simpler operations of addition, subtraction, multiplication, and division, respectively.

The $\mathfrak{L}$-transformation method is used to simplify the solution of linear constant-coefficient integrodifferential equations (and other types of equations) in an analogous way. That is, by applying the direct transformation to the equation and its initial conditions there is obtained a simpler equation. In fact, it is an algebraic equation rather than an i-d equation. This simpler equation is then solved for an intermediate function, from which the desired solution of the original equation is obtained by the application of the inverse $\mathfrak{L}$ transformation. In practice, the direct and inverse transformations are effected by a table which is used both from left to right and from right to left.

It will appear that there are essential differences between the logarithmic and the $\mathfrak{L}$ transformations. One in particular may be mentioned. The logarithmic transformation is used to transform numbers into other numbers, whereas the $\mathfrak{L}$ transformation is used to transform functions into other functions. The basic methods are quite analogous, however, both transformations having in common the important property of transforming certain operations into simpler operations. This property constitutes the fundamental reason for their use.

For those who are not already acquainted with the $\mathfrak{L}$ transformation and its inverse, this chapter will introduce these transformations through

a comparison with Fourier series and integrals. This approach is chosen because those who are interested in the $\mathfrak{L}$-transformation method frequently have some previous knowledge of the Fourier series. From the series follow naturally the Fourier-integral transformation and then the $\mathfrak{L}$ transformation. Another reason for the approach chosen here is the comparison which it affords of the Fourier and Laplace integrals — a comparison which would be desirable regardless of the approach.

2. Periodic and Nonperiodic Functions

At the outset it is important to have clearly in mind the distinction between a periodic and a nonperiodic function. For t a real variable, a function $p(t)$ is called *periodic* with period T if there is a real number T such that $p(t \pm T) = p(t)$ for $-\infty < t < \infty$. If a function of t does not satisfy this definition it is said to be *nonperiodic*. For example, with a, b, and T real numbers, the function

$$p(t) \triangleq a \cos \frac{2\pi}{T} t + b \sin \frac{2\pi}{T} t, \qquad -\infty < t < \infty$$

is periodic with period T and angular frequency $2\pi/T$, whereas the function

$$f(t) \triangleq \begin{array}{ll} 0, & t < 0 \\ a \cos \dfrac{2\pi}{T} t + b \sin \dfrac{2\pi}{T} t, & 0 < t \end{array}$$

is nonperiodic. It will be observed that at its point of discontinuity $f(t)$ has been left undefined. This same practice will be followed for functions introduced later. The reason for this will be given later in Chapter 4, Sec. 7.

Here, and in much of the mathematical treatment that follows, the symbol t is used for the independent real variable because in the applications of the theory, time is usually the independent variable. This symbol, however, may be considered to represent any independent real variable, as for instance, a space coordinate.

The periodic function of the above example is sinusoidal. It can be written as a cosine with an initial phase angle, i.e.,

$$p(t) \triangleq |P| \cos \left(\frac{2\pi}{T} t + \psi \right)$$

in which $|P| \triangleq (a^2 + b^2)^{1/2}$ and $\psi \triangleq \tan^{-1} (-b/a)$. It can be resolved by Euler's formula into the sum of two complex exponential functions

as follows:

$$p(t) \triangleq |P| \cos\left(\frac{2\pi}{T}t + \psi\right)$$

$$= \tfrac{1}{2}(|P|e^{j\psi}e^{j\frac{2\pi}{T}t} + |P|e^{-j\psi}e^{-j\frac{2\pi}{T}t}),$$

$$= \tfrac{1}{2}(Pe^{j\frac{2\pi}{T}t} + \bar{P}e^{-j\frac{2\pi}{T}t}), \qquad [1]$$

in which $j \triangleq \sqrt{-1}$, $P \triangleq |P|\, e^{j\psi}$, and $\bar{P} \triangleq |P|e^{-j\psi}$. Here P and $\bar{P}$ are the conjugate complex amplitudes of the exponential functions of t composing $p(t)$, and ψ is the angle of initial phase difference with respect to a chosen reference line. Observe that the angular frequency $2\pi/T$ of the cosine appears in the exponents of both exponentials — in one with a positive sign and in the other with a negative sign. These exponentials, which are complex functions of the real variable t, are not individually interpretable in physical terms. They form, nevertheless, the basis of the representation of sinusoidal functions by rotating plane-vectors which have played such an important role in the treatment of steady-state sinusoidal oscillations in physical systems.

In steady-state calculations it is customary to use complex numbers in place of sinusoidal functions. The substitution of the complex number P for the sinusoidal time function $|P| \cos\left(\frac{2\pi}{T}t + \psi\right)$ is in a sense a functional transformation. To indicate this transformation, it can be written $\mathcal{T}[p(t)] = P$ and its inverse $\mathcal{T}^{-1}[P] = p(t)$. Periodic functions are transformed into complex numbers because the operations of addition, subtraction, multiplication, and division are carried out more easily with complex numbers than with periodic functions.

With the introduction of complex functions such as $e^{j\frac{2\pi}{T}t}$ it becomes necessary to consider functions of complex variables and complex functions of real variables. This will be done in more detail in the next chapter.

3. Fourier Series

Consider next a function $p(t)$ which is periodic but nonsinusoidal. A portion of such a function is shown in Fig. 3·1.

Under certain conditions a function can be expanded in a Fourier series which converges for all points at which the function is continuous. Written in the familiar trigonometric form the Fourier series which

represents such a function $p(t)$ with period T is

$$p(t)\ (=)\ \frac{a_0}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\left(a_n \cos\frac{n2\pi}{T}t + b_n \sin\frac{n2\pi}{T}t\right), \qquad [2]$$

in which n is a positive integer, and the coefficients are computed from the integrals

$$\begin{aligned} a_0 &\triangleq \int_{-T/2}^{T/2} p(t)dt, \\ a_n &\triangleq \int_{-T/2}^{T/2} p(t)\cos\frac{n2\pi}{T}t\,dt, \\ b_n &\triangleq \int_{-T/2}^{T/2} p(t)\sin\frac{n2\pi}{T}t\,dt. \end{aligned} \qquad [3]$$

In equation 2 the symbol $(=)$ means *equals almost everywhere.* "Almost everywhere" is a precise mathematical expression meaning "everywhere except for a set of points (here representing values of t) which can be covered by a set of line segments the sum of whose lengths is arbitrarily small." Such a set might contain infinitely many points. Equals almost everywhere implies in particular "equals at all points of continuity" [Ho 1].

The factor $1/T$ which usually appears in the equations for the coefficients a_0, a_n, and b_n of equation 3 has here been transferred to equation 2 in order to facilitate comparisons that will be made later in Secs. 4 and 6.

The conditions alluded to on page 96 for the representation of a function by a Fourier series insure the convergence of the coefficient integrals 3 and the convergence of the series 2 to $p(t)$. In other words, it must be possible to compute the coefficients and to find the sum of the series, and the series must converge at points of continuity to the function.

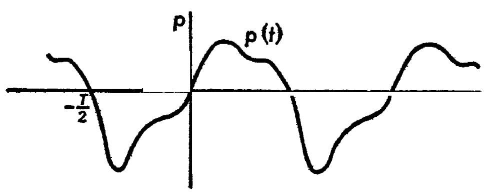


FIG. 3·1. A periodic but nonsinusoidal function.

The complex-exponential form of the series can be written more compactly than form 2 and is often more convenient to use. To derive it, each pair of single-frequency terms in equation 2 can be written as the sum of two complex exponentials, as shown above for the sinusoidal

function. Thus the typical pair,

$$a_n \cos \frac{n2\pi}{T} t + b_n \sin \frac{n2\pi}{T} t$$

$$= \tfrac{1}{2} [(a_n - jb_n)e^{j\frac{n2\pi}{T}t} + (a_n + jb_n)e^{-j\frac{n2\pi}{T}t}].$$

By use of equations 3 and Euler's formula,

$$a_n - jb_n = \int_{-T/2}^{T/2} p(t)e^{-j\frac{n2\pi}{T}t}dt \triangleq P\left(\frac{n2\pi}{T}\right). \qquad [4]$$

If now the angular frequency $n2\pi/T$ is set equal to ω, the typical pair of terms can be written as

$$\tfrac{1}{2} [P(\omega)e^{j\omega t} + P(-\omega)e^{-j\omega t}].$$

Note that $P(-\omega) = \overline{P}(\omega)$. $P(\omega)$ and $P(-\omega)$ are conjugate complex amplitude coefficients. They are called "coefficients" because they represent only the *relative* magnitudes and initial phases of the actual complex amplitudes which are $2/T$ times these coefficients.

If n is extended to include zero and negative integers, it is observed from equation 4 that $P(0) = a_0$, and the trigonometric form of the Fourier series given in equation 2 can be written

$$p(t) \;(=)\; \frac{1}{T} \sum_{\omega=-\infty}^{\omega=\infty} P(\omega)e^{jt\omega}. \qquad [5]$$

But $1/T = \omega/2\pi n$, so equation 5 can be written

$$p(t) \;(=)\; \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} P(\omega)e^{jt\omega} \frac{\omega}{n}, \qquad [6]$$

and equation 4 becomes

$$P(\omega) = \int_{-T/2}^{T/2} p(t)e^{-j\omega t}dt. \qquad [7]$$

It will be observed that in this form a single sum 6 and a single coefficient integral 7 suffice.

Equations 7 and 6 are functional transformations and can be written

$$\mathcal{F}_s[p(t)] = P(\omega), \qquad [7']$$

and

$$\mathcal{F}_s^{-1}[P(\omega)] \;(=)\; p(t). \qquad [6']$$

By means of equation 7 a complex function of ω is obtained which is easier to handle algebraically than the real function of t which it represents. This is an extension to a complicated periodic function of the

advantages gained when a complex number is substituted for a simple sinusoidal function as stated on page 96.

4. Fourier Integral Transformation

In the foregoing discussion only periodic functions have been considered. Nonperiodic functions will be considered next. An example of such a function is shown in Fig. 3·2.

Under certain conditions a nonperiodic function $f(t)$ can be represented in a finite range by a Fourier series which is convergent almost everywhere.[1] The origin can be chosen, as in Fig. 3·2, at the midpoint of the selected range and the time values $-T/2$ and $T/2$ assigned to the ends of the range. This interval is chosen as the period T of an artificial periodic function which repeats, in equal intervals in each direction, the same form as found in the interval $-T/2 \leqq t \leqq T/2$. The expansion of this artificial periodic function in a Fourier series, with coefficients found from the values of the function $f(t)$ in the interval chosen, will yield a series which in general will converge to the function $f(t)$ in this chosen interval only.

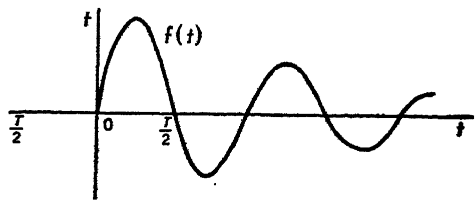


FIG. 3·2. A nonperiodic function.

Carrying this treatment of nonperiodic functions further, consider a nonperiodic function which is representable by a convergent Fourier series almost everywhere over *every* finite range. It may then be asked: (1) What becomes of the Fourier series that represents a chosen range of the function as this range is extended indefinitely in both directions? (2) What happens to the integral giving the coefficients?

The answers to these questions are: (1) With certain additional restrictions on the growth of the function as $|t| \to \infty$, formally the sum in the series becomes an integral

$$f(t) \ (=) \ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{jt\omega} d\omega, \tag{8}$$

which is called a Fourier integral [Wi 5]. (2) The coefficients merge into a continuously defined function; the limits on the coefficient integral become infinite; and this integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \tag{9}$$

[1] For meaning of "almost everywhere" see page 97.

which is also called a Fourier integral, now provides the coefficient function. Equation 8 should be compared with equation 6 and 9 with 7. $F(\omega)$ is the complex amplitude "coefficient" — it represents only the *relative* magnitudes and initial phases of the actual complex amplitudes, these amplitudes being infinitesimals.

Since the concern here is with a nonperiodic function, the range of integration must be the entire range of definition of the function, i.e., $-\infty < t < \infty$, rather than merely a typical period as in the case of periodic functions.

The integral transformations given in equations 9 and 8 are called *Fourier transformations.* They can be written

$$\mathfrak{F}[f(t)] = F(\omega) \tag{9'}$$

and

$$\mathfrak{F}^{-1}[F(\omega)]\ (=)\ f(t). \tag{8'}$$

The one given in equation 9 will be called the *direct Fourier transformation;* the one given in 8 will be called the *inverse Fourier transformation.* These transformations are functional transformations since they transform a function of the variable t into what is in general an entirely different function of the variable ω, and vice versa. The advantages that result from the replacement of a complicated function of a real variable by a relatively simple function of the angular frequency are in this way extended to nonperiodic functions.

5. Unilateral Fourier Transformation

Among the driving functions most commonly used in problems whose solution is simplified by the transformation method are the unit step function

$$u(t) \triangleq \begin{cases} 0 & t < 0, \\ 1 & 0 < t, \end{cases}$$

shown in Fig. 3·3, and the unit sinusoid section $\sin(\beta t + \psi)\cdot u(t)$ shown in Fig. 3·4. Neither of these nonperiodic functions can be handled by the $\mathfrak{F}$ transformation 9 without an additional limit process. Any attempt to compute the $\mathfrak{F}$ transform of either leads to an improper integral in the sense that it is not convergent, i.e., the limit process implied by the integral does not lead to a finite limit. Since these two functions, which are examples of the most elementary types of driving functions, cannot be transformed directly by the $\mathfrak{F}$ transformation 9, it follows that a transformation capable of handling these and more general functions is needed. The Laplace transformation fills these require-

ments. The Fourier transformation 9 may be generalized to fill them also. Attention is directed now to these forms, the generalization of the $\mathcal{F}$ transformation being treated first.

One way to extend the range of applicability of the transformation 9 is to multiply the function to be transformed by a convergence factor e^{-ct}, $0 < c$, so that the product function will decrease as $t \to \infty$ and make the integral converge. Notice, however, that if the product of the function to be transformed and the convergence factor does not approach zero for $t \to -\infty$, such a factor becomes a divergence factor for $t \to -\infty$. This mathematical difficulty can be overcome by shifting the lower limit of integration in equation 9 from $-\infty$ to 0. That this sectioning of the range of integration is *permissible* without loss, so far as application to problems treated in this text is concerned, follows from the fact that in these problems interest centers on what happens *after* a particular instant of time, i.e., one is looking forward in time. For example, one's interest may begin at the instant at which a switch closes an electric circuit. By choosing the origin of the time coordinate at this special instant, attention may be confined to what happens for only nonnegative values of the time variable. Later, in Chapter 5, it will be seen that cutting off the range of integration is *necessary* if the trans-

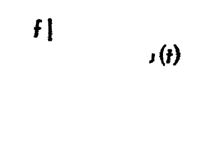


FIG. 3-3. Unit step function.

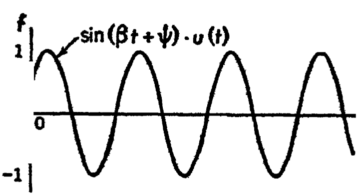


FIG. 3-4. Unit sinusoid section.

formation is to bring in naturally the boundary conditions at the origin. Because of the one-sided nature of the transformation remaining after the range of integration has been reduced to that from 0 to ∞, the transformation is called unilateral.

Although the point of view taken here is that the transformation is cut off to the left of the origin, it is also possible for certain purposes to take the point of view that the transformation range of integration is unchanged, but that all the functions to be transformed are cut off to the left of the origin. Where, as in most cases, it is desired to bring in

naturally the boundary conditions at the origin the first point of view is the preferable one to take.

More specifically, it can be stated that if the function $f(t)$ is single valued almost everywhere for $0 \leqq t$, and if there is a real number c such that

$$\lim_{T\to\infty} \int_0^T |f(t)| e^{-ct} dt < \infty, \tag{10}$$

then the unilateral $\mathfrak{F}$ transformation can be used to find the transform of $f(t)e^{-ct}$ for $0 \leqq t$. Under these conditions the direct transformation can be written

$$\int_0^\infty [f(t)e^{-ct}]e^{-j\omega t} dt = F(c,\omega). \tag{11}$$

With $f(t)$ specified, the *abscissa of absolute convergence* of integral 11 is defined as the greatest of the lower bounds of the set of numbers c that satisfy equation 10. It will be denoted by σ_a. Since σ_a depends upon $f(t)$, for brevity it will be convenient to speak of "the σ_a of $f(t)$." Absolute convergence is required rather than merely conditional convergence so that later the order of performing certain limit processes may be changed, and in particular so that the strong form of multiplication theorem can be used [AM 1, Do 15].

In this treatment attention will be confined to only those functions whose $\sigma_a < +\infty$. By considering a function's form, usually it is not difficult to find the value of its σ_a. The examples in Table 2 illustrate this. A formula [LA 1a] for σ_a will be given in Volume 2.

TABLE 2. ABSCISSA OF ABSOLUTE CONVERGENCE

$f(t)$	Inequality 10 holds for	σ_a for Integral 11
1	$0 < c$	0
$u(t)$	$0 < c$	0
$\sin \beta t$	$0 < c$	0
$\sin \beta t \cdot u(t)$	$0 < c$	0
$e^{-\alpha t}, 0 < \alpha$	$-\alpha < c$	$-\alpha$
$e^{\alpha t}, 0 < \alpha$	$\alpha < c$	α
t^2	$0 < c$	0
$u(t-a) - u(t-b), a < b$	$-\infty < c$	$-\infty$
e^{t^2}	No value for c	No value for σ_a

From Table 2 it may be observed that, for the functions shown, the σ_a is unchanged if the portion of the function lying to the left of the origin is cut off. This is a result of the one-sidedness of the transformation, which has zero as its lower limit.

So far nothing has been said regarding the inverse of the unilateral direct transformation 11. This inverse is bilateral. In fact, it is the same inverse 8 as that for the bilateral direct transformation 9. Application of the $\mathcal{F}^{-1}$ transformation to the unilateral transform $F(c,\omega)$ gives as a result the product $f(t)e^{-ct}$ for $0 \leqq t$. Accordingly the unilateral transformation and its inverse are written

$$\int_0^\infty [f(t)e^{-ct}]e^{-j\omega t}dt = F(c,\omega), \qquad \sigma_a < c, \tag{12}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(c,\omega)e^{jt\omega}d\omega \ (=) \ f(t)e^{-ct}, \qquad 0 \leqq t. \tag{13}$$

The forms 12 and 13 can be used with the extra limit process of letting $c \to 0$ to extend the range of applicability of the $\mathcal{F}$ transformation to a few functions $f(t)$ which would cause the improper integral in 12 with $c = 0$ to diverge because these functions $f(t)$ do not decrease rapidly enough. This method of finding transforms works, for example, on a function such as $u(t)$ but fails with one such as $e^{\alpha t}$, $0 < \alpha$. The reason for this can be better appreciated after a discussion of the relation between the form of the time function and the positions in the complex plane of the singularities of its transform. Although this use of the $\mathcal{F}$ transformation with a convergence factor and subsequent limit process works in many cases, it is at best a circuitous procedure for extending the range of applicability of the transformation.

By slight modification, equations 12 and 13 give what is called a generalized (or complex) form of $\mathcal{F}$ transformation. Thus by associating the convergence factor e^{-ct} with the kernel $e^{-j\omega t}$ of the transformation instead of with the function $f(t)$ to be transformed, equation 12 becomes

$$\int_0^\infty f(t)e^{-(c+j\omega)t}dt = F(c + j\omega), \qquad \sigma_a < c. \tag{14}$$

The corresponding generalization in the inverse transformation can be made by multiplying both sides of equation 13 by the factor e^{ct} and changing the variable of integration from ω to $c + j\omega$. There results

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(c + j\omega)e^{t(c+j\omega)}d(c + j\omega) \ (=) \ f(t),$$
$$0 \leqq t, \qquad \sigma_a < c. \tag{15}$$

This generalization of the $\mathcal{F}$ transformation and its inverse leads naturally to the Laplace transformation by letting c be a real variable σ

instead of a real constant. The Laplace transformation is thus the result of one further step in generalization. Because of this generalization it has an even wider range of application than the complex $\mathfrak{F}$ transformation which it includes. But more important, by its use the complication of the ordinary $\mathfrak{F}$ transformation, with the extra limit process which is needed in many practical cases, is avoided.

6. Laplace Transformation

Although actually some 30 years older [Appen C] than the Fourier transformation, the Laplace transformation is not so widely known.

The complex variable $\sigma + j\omega$ of the Laplace transformation will be abbreviated to the single letter s. It replaces the complex variable $c + j\omega$ in equations 14 and 15. The restriction on c in equation 10 now passes to σ. Equation 14 for the complex amplitude coefficient becomes

$$\int_0^{\infty} f(t)e^{-st}dt = F(s), \qquad \sigma_a < \sigma. \tag{16}$$

This unilateral form of the Laplace transformation is but one of many forms. Others important for certain other uses are the bilateral and the Stieltjes forms. The form used in this text seems to be the simplest one which will handle the problems treated here. Equation 15 becomes

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{ts}ds \,(=)\, f(t), \qquad 0 \leqq t, \qquad \sigma_a < c. \tag{17}$$

The constant c is retained in the limits of equation 17 to indicate a straight-line integration path paralleling the axis of imaginaries.

The transformation 16 is called the *direct Laplace transformation* (abbreviated $\mathfrak{L}$ *transformation*). It can be appreciated that for the integral 16 to have a simple interpretation, $f(t)$ must be single valued almost everywhere in the range $0 \leqq t$ and must not grow so rapidly as $t \rightarrow \infty$ that the convergence factor, which has now been absorbed in the kernel e^{-st}, will be inadequate. This will be more apparent if it is recalled that

$$e^{-st} \equiv e^{-\sigma t}e^{-j\omega t} \equiv e^{-\sigma t}\,(\cos \omega t - j \sin \omega t),$$

and 16 is rewritten in the expanded form

$$\int_0^{\infty} f(t)e^{-st}dt \equiv \int_0^{\infty} f(t)e^{-\sigma t}\cos \omega t dt - j\int_0^{\infty} f(t)e^{-\sigma t}\sin \omega t dt. \tag{18}$$

One should note also that $F(s)$ is defined by transformation 16 only in the region of absolute convergence of the integral, i.e., in the part of the complex (s) plane (called a *half-plane* in mathematics) for which $\sigma_a < \sigma$.

For the *inverse of the Laplace transformation* (abbreviated $\mathfrak{L}^{-1}$ *transformation* and given in equation 17) the number c is any real number greater than the σ_a belonging to $f(t)$ and its transform $F(s)$. Accordingly, the path of integration in the complex plane is generally a line through the point $\sigma = c$, paralleling the axis of imaginaries as shown in Fig. 3·5. The evaluation of this integral by integration in the complex plane will be treated in Volume 2. The resolution of functions by the $\mathfrak{L}^{-1}$ transformation is in terms of infinitesimal exponentially damped oscillations, only the relative magnitudes and phases of which are represented by $F(s)$.

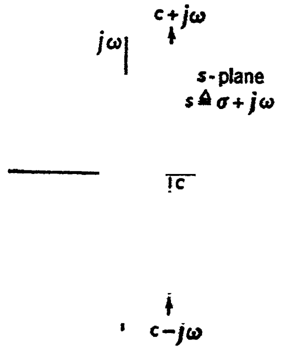


FIG. 3·5. Path of integration represented by the limits on the $\mathfrak{L}^{-1}$-transformation integral.

The transformations $\mathfrak{L}$ and $\mathfrak{L}^{-1}$, like $\mathcal{F}$ and $\mathcal{F}^{-1}$, are functional transformations. The *result* of the direct transformation is called the *direct Laplace transform* (abbreviated $\mathfrak{L}$ *transform*), whereas the *result* of the inverse transformation is called the *inverse Laplace transform* (abbreviated $\mathfrak{L}^{-1}$ *transform*). In symbols, equation 16 can be abbreviated to

$$\mathfrak{L}[f(t)] = F(s), \qquad \sigma_a < \sigma, \tag{16'}$$

and 17 to

$$\mathfrak{L}^{-1}[F(s)] \ (=) \ f(t), \qquad 0 \leq t. \tag{17'}$$

The sequence of forms in the foregoing progressive generalization from the Fourier series and coefficient integral to the Laplace integrals is given in Table 3.

It may be noted from Table 3 that in all cases the inverse transformations are bilateral, i.e., the variable runs over a doubly infinite range. This is true even where the corresponding direct transformation is unilateral.

Furthermore, it may be observed that if $\sigma_a < 0$, and $f(t) = 0$ for $t < 0$, the $\mathfrak{L}$ transformation reduces to the bilateral $\mathcal{F}$ transformation. Thus under these conditions the $\mathfrak{L}$ transformation and its inverse become

$$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = F(j\omega),$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{jt\omega}d\omega \ (=) \ f(t).$$

NVERSE TRA ATIONS

Type	Direct Transformation	Inverse Transformation
$\mathcal{F}$ series	$\int_{-T/2}^{T/2} p(t)e^{-j\omega t}dt = P(\omega), \quad \omega \triangleq \frac{n2\pi}{T}$	$\frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} P(\omega)e^{jt\omega}\frac{\omega}{n} (=) p(t)$
Bilateral $\mathcal{F}$ integral	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = F(\omega)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{jt\omega}d\omega (=) f(t)$
Unilateral $\mathcal{F}$ integral	$\int_0^{\infty} [f(t)e^{-ct}]e^{-j\omega t}dt = F(c,\omega), \quad \sigma_a < c$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(c,\omega)e^{jt\omega}d\omega (=) f(t)e^{-ct} \quad 0 \leqq t, \quad \sigma_a < c$
Complex $\mathcal{F}$ integral	$\int_0^{\infty} f(t)e^{-(c+j\omega)t}dt = F(c+j\omega), \quad \sigma_a < c$	$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(c+j\omega)e^{t(c+j\omega)}d(c+j\omega) (=) f(t),$ $0 \leqq t, \quad \sigma_a < c$
$\mathfrak{L}$ integral	$\int_0^{\infty} f(t)e^{-st}dt = F(s), \quad \sigma_a < \sigma$	$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{ts}ds (=) f(t), \quad 0 \leqq t, \quad \sigma_a < c$

Here the direct transform $F(j\omega)$ is written with the imaginary argument $j\omega$ rather than the usual real argument ω because of an unessential change in point of view.

In the next chapter several basic theorems on the $\mathfrak{L}$ transformation and its inverse will be stated and their use illustrated. Later, in Volume 2, other forms of the direct and inverse transformations will be discussed.

PROBLEMS

3-1. (a) Determine the complex amplitude coefficient $P(\omega)$ (i.e., the $\mathfrak{F}_s$ transform) for the periodic function $p(t)$ shown in the diagram and make plots of the

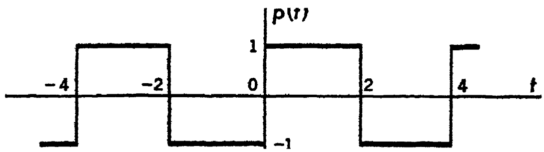


FIG. 3-P1

magnitude and the initial phase of $P(\omega)$ versus ω. Keep the magnitude function positive and let the phase function account for any negative signs.

(b) Repeat part a making the period $T = 8$. Plot the results to the same scale as used in a and describe the effect of doubling the period of $p(t)$.

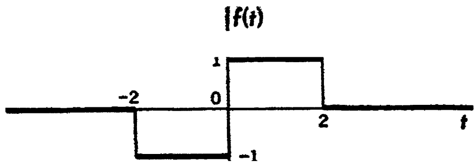


FIG. 3-P2

3-2. Determine the complex amplitude coefficient $F(\omega)$ (i.e., the $\mathfrak{F}$ transform) for the nonperiodic function $f(t)$ shown in the diagram and plot the magnitude and initial phase of $F(\omega)$ versus ω.

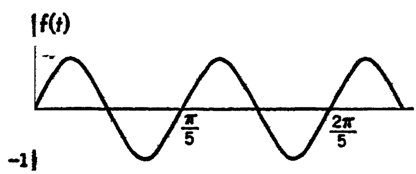


FIG. 3-P3

3-3. Determine the complex amplitude coefficient $F(s)$ (i.e., the $\mathfrak{L}$ transform) for the nonperiodic function $f(t) \triangleq \sin 10t$, $0 \leq t$, shown in the diagram and plot the magnitude and initial phase of $F(s)$ versus s for (a) $\sigma = 1$ and (b) $\sigma = 5$.

CHAPTER IV

THE $\mathfrak{L}$ TRANSFORMATION AND ITS APPLICATION TO SIMPLE FUNCTIONS

A. INTRODUCTION TO THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

Chapter 3 introduced the $\mathfrak{L}$ transformation and its inverse. There it was shown that the $\mathfrak{L}$ transform of a function f of a real variable t is a function F of a complex variable s. To provide a basis for the presentation of theorems on the $\mathfrak{L}$ transformation and transforms in this chapter, and for similar treatment of the integral $\mathfrak{L}^{-1}$ transformation in Volume 2 it is convenient to begin with certain definitions and ideas from the elementary theory of functions of a complex variable [Cu 1].

1. Complex Plane; Functions of a Complex Variable

The values of the complex variable $s \triangleq \sigma + j\omega$ with its component real and imaginary parts, the real variables σ and ω, can be represented geometrically by the points in a plane. This plane is called the *complex plane*, or *s-plane*. The variables σ and ω are treated as rectangular coordinates, the axis of abscissas being used as the real or σ-axis, and the axis of ordinates as the imaginary or $j\omega$-axis. It is sometimes convenient to call the $\lim\limits_{s\to\infty} s$, regardless of the path traversed, the *point at infinity*. For a geometric representation of the values of s in such cases a sphere is used rather than a plane, the origin being chosen at one pole and the point at infinity at the other.

Let G be an unrestricted function of s. In general, G will be a complex function, so it may be written

$$G(s) \triangleq U(\sigma,\omega) + jV(\sigma,\omega), \qquad [1]$$

in which U is its real part and V is its imaginary part. G can be represented geometrically in a complex G-plane in the same way that s is represented in the complex s-plane. In the G-plane U is measured along the axis of reals and V along the axis of imaginaries.

Example 1. Plot $G_1(s) \triangleq \dfrac{1}{s+a}$ for a variation of s along the imaginary axis from $-j\infty$ to $j\infty$.

Here $\sigma = 0$, and

$$G_1(s) = \frac{1}{j\omega + a} = \frac{a}{a^2 + \omega^2} + j\frac{-\omega}{a^2 + \omega^2} \triangleq U(\omega) + jV(\omega). \qquad [2]$$

Figure 4·1 shows the paths followed by s and G in their respective planes.

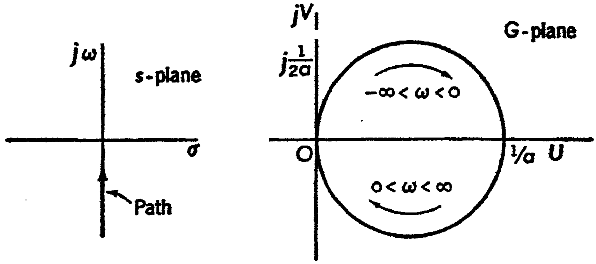


FIG. 4·1. Plot of $G_1(s) \triangleq (s + a)^{-1}$ for a variation of s along the axis of imaginaries.

Example 2. Plot $G_2(s)$, with $[G_2(s)]^2 \triangleq \dfrac{1}{s + a}$, for a variation of s along the imaginary axis from $-j\infty$ to $j\infty$.

Again $\sigma = 0$, and

$$G_2(s) = \pm\frac{1}{(j\omega + a)^{\frac{1}{2}}} = \pm\frac{1}{[(a^2 + \omega^2)^{\frac{1}{2}}e^{j\phi(\omega)}]^{\frac{1}{2}}}$$

$$= \pm\left[\frac{\cos\dfrac{\phi(\omega)}{2}}{(a^2 + \omega^2)^{\frac{1}{4}}} + j\frac{-\sin\dfrac{\phi(\omega)}{2}}{(a^2 + \omega^2)^{\frac{1}{4}}}\right] \qquad [3]$$

in which $\phi(\omega) \triangleq \tan^{-1} \omega/a$. $G_2(s)$ thus has two values for each value of s. Figure 4·2 shows the paths of s and G in their respective planes.

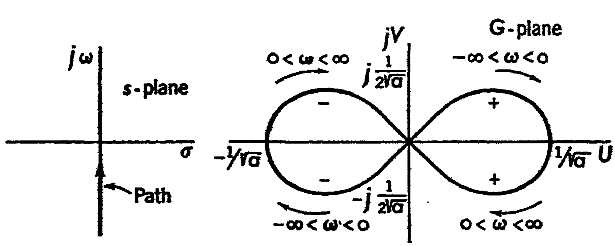


FIG. 4·2. Plot of $[G_2(s)]^2 \triangleq (s + a)^{-1}$ for a variation of s along the axis of imaginaries.

In equation 1, $G(s)$ was given in rectangular form $U(\sigma,\omega) + jV(\sigma,\omega)$. It can also be expressed in polar form $R(\sigma,\omega)e^{j\Phi(\sigma,\omega)}$, in which R is the radius or magnitude function and Φ is the phase function. Since R is

taken as positive, negative signs are taken care of by Φ. Magnitude and phase functions are used extensively in showing graphically the steady-state characteristics of electric and mechanical systems.

Example 3. Plot the magnitude and phase functions of $G_1(s) \triangleq \dfrac{1}{s+a}$ corresponding to the variation of s along the imaginary axis from $-j\infty$ to $j\infty$.

As in Example 1, $\sigma = 0$, and

$$G_1(s) = \frac{1}{j\omega + a} = \frac{1}{(a^2+\omega^2)^{\frac{1}{2}}} e^{j\tan^{-1}(-\omega/a)} \triangleq R(\omega)e^{j\Phi(\omega)} \qquad [2']$$

The magnitude and phase functions are shown in Fig. 4·3. They are related to the curve for this same G-function shown in Fig. 4·1 as follows: $R(\omega)$ is the length of the radius vector, and $\Phi(\omega)$ is the angle this radius vector makes with the U-axis.

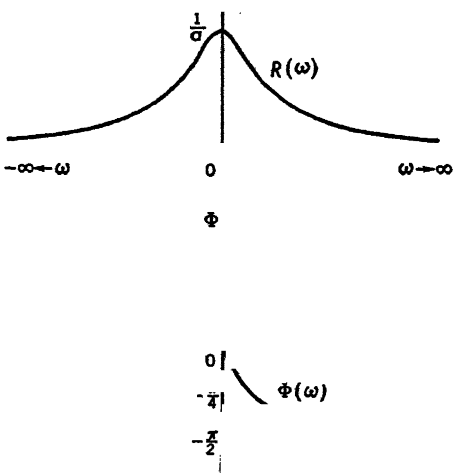


FIG. 4·3. Magnitude and phase functions of $G_1(s) \triangleq (s+a)^{-1}$ for a variation of s along the axis of imaginaries.

2. SINGLE-VALUED FUNCTIONS; CONTINUITY; DERIVATIVE

$G(s)$ is a *single-valued function* of s if to each value of s there corresponds only one value of G. $G_1(s)$ in Ex. 1 is an example of a single-valued function, whereas $G_2(s)$ in Ex. 2 is an example of a double-valued function.

$G(s)$ is *continuous* at the point s_1 if

$$\lim_{\Delta s \to 0} G(s_1 + \Delta s) = G(s_1) \qquad [4]$$

regardless of the set of (complex) values through which Δs approaches zero.

If

$$\lim_{\Delta s \to 0} \frac{G(s_1 + \Delta s) - G(s_1)}{\Delta s} = G'(s_1) \qquad [5]$$

regardless of the set of (complex) values through which Δs approaches zero, then $G'(s_1)$ is the *derivative* of G with respect to s at s_1. It is also written $\dfrac{dG(s)}{ds}\cdot$

3. Analytic Functions

If a function $G(s)$ has a unique derivative $G'(s_1)$ at the point s_1 in the complex plane, the function is said to be *analytic* at the point s_1. The Cauchy-Riemann necessary and sufficient condition that a function be analytic will be found in texts on functions of a complex variable [Cu 1].

A function that is analytic at every point in a region of the complex plane is by way of abbreviation said to be analytic in this region. Further, it can be shown [Bi 1] that if a function of a complex variable possesses a first derivative inside a simple contour, then it also possesses derivatives of arbitrarily high order here, and these are all analytic functions.

Example 1. Consider the function $G_3(s) \triangleq s = \sigma + j\omega$.
Here the

$$\lim_{\Delta s \to 0} \frac{G_3(s + \Delta s) - G_3(s)}{\Delta s} = \lim_{\substack{\Delta\sigma \to 0 \\ \Delta\omega \to 0}} \frac{(\sigma + \Delta\sigma) + j(\omega + \Delta\omega) - (\sigma + j\omega)}{\Delta\sigma + j\Delta\omega}.$$

If the limit process is carried out by first letting $\Delta\sigma \to 0$ and then letting $\Delta\omega \to 0$ there is obtained

$$\lim_{\Delta\omega \to 0} \frac{\sigma + j(\omega + \Delta\omega) - (\sigma + j\omega)}{j\Delta\omega} = 1. \qquad [6]$$

If instead, the limit process is carried out by first letting $\Delta\omega \to 0$ and then letting $\Delta\sigma \to 0$, there is obtained

$$\lim_{\Delta\sigma \to 0} \frac{(\sigma + \Delta\sigma) + j\omega - (\sigma + j\omega)}{\Delta\sigma} = 1. \qquad [7]$$

Furthermore, it would be found that the limit would be 1 regardless of how $\Delta s \to 0$. The function s can be shown to be analytic everywhere in the finite part of the complex plane.

Example 2. Show that $G_4(s) \triangleq \bar{s} = \sigma - j\omega$ is not an analytic function. ($\bar{s}$ is read " s conjugate." The *conjugate* of a function of a complex variable is obtained by replacing every j in the function by $-j$.)

Here the

$$\lim_{\Delta s \to 0} \frac{G_4(s + \Delta s) - G_4(s)}{\Delta s} = \lim_{\substack{\Delta \sigma \to 0 \\ \Delta \omega \to 0}} \frac{(\sigma + \Delta\sigma) - j(\omega + \Delta\omega) - (\sigma - j\omega)}{\Delta\sigma + j\Delta\omega}.$$

If Δs is allowed to approach zero by first letting $\Delta\sigma \to 0$, and then letting $\Delta\omega \to 0$, the limit of the difference quotient is

$$\lim_{\Delta\omega \to 0} \frac{\sigma - j(\omega + \Delta\omega) - (\sigma - j\omega)}{j\Delta\omega} = -1. \qquad [8]$$

If $\Delta s \to 0$ by first letting $\Delta\omega \to 0$ and then letting $\Delta\sigma \to 0$, the limit of the difference quotient is

$$\lim_{\Delta\sigma \to 0} \frac{(\sigma + \Delta\sigma) - j\omega - (\sigma - j\omega)}{\Delta\sigma} = 1. \qquad [9]$$

Since the limit of the difference quotient is not independent of the way in which $\Delta s \to 0$, the function $\bar{s}$ is not analytic.

4. Zeros; Singular Points

If the function $G(s)$ can be expressed as

$$G(s) = (s - s_1)^m G_a(s), \qquad [10]$$

in which m is a positive integer, and $G_a(s_1)$ is finite and different from zero, then $G(s)$ is said to have a *zero of order m* at the point s_1.

Points of the s-plane at which a function $G(s)$ is not analytic (i.e., does not possess a unique derivative) are called *singular points* of G.

The simplest type of singular point is a *pole*. If the function $G(s)$ can be expressed as

$$G(s) = \frac{G_b(s)}{(s - s_1)^m}, \qquad [11]$$

in which m is a positive integer, and $G_b(s_1)$ is finite and different from zero, then $G(s)$ is said to have a *pole of order m* at the point s_1.

Relation 11 might be rewritten as

$$(s - s_1)^m G(s) = G_b(s). \qquad [12]$$

In this form it can be interpreted as stating that if $G(s)$ has an mth-order pole at s_1, this pole can be removed by multiplying $G(s)$ by the mth power of the linear factor $(s - s_1)$.

As an example of a function having zeros and poles, consider the rational fraction

$$G_5(s) \triangleq \frac{(s + 1)^2}{s(s + 2)^3}. \qquad [13]$$

This function has a second-order zero at -1, a first-order pole at 0, and a third-order pole at -2.

A single-valued function $G(s)$ has an *essential singularity* at point s_1 if it is impossible to reduce $G(s)$ as in equation 12 to an analytic function $G_b(s)$ at this point by multiplying it by $(s - s_1)^m$, with m some finite positive integer. As examples, $G_6(s) \triangleq e^s$ has an essential singularity at ∞, and $G_7(s) \triangleq e^{\frac{1}{s+a}}$ has an essential singularity at $-a$.

5. Extension of Region of Definition of a Function

Frequently in the treatment that follows, a function of a complex variable is defined in a restricted region of the s-plane by an integral. The limitation on the region of definition of the function arises from difficulties with the integral defining the function. An extension of this region of definition therefore cannot be made through this integral. In this region of definition the function can often be expressed in closed form, for example, as a rational function. If so, an extension of this region of definition can be made by application of the principle of *extension through preservation of form* of the function.

It is natural, if the range of definition is to be extended more or less by fiat, that one should choose to preserve the *form* of the function in the new range. Fortunately, an extension of the region of definition by preservation of form of the function turns out to be the most convenient way of extending this region for ordinary purposes. Furthermore, the function so extended will be identical with that which would be obtained by the method of analytic continuation [Cu 1, Bi 1] in the region in which the function can be analytically continued, i.e., at all points except where it is barred by singularities.

This preservation-of-form method of extending the region of definition of a function beyond the boundary where it is defined by an integral has been applied to the gamma function in extending its definition into the left half-plane where there are singularities at the negative integers.

Example 1. Let the function $1/s$ be defined by an integral in the half-plane in which $0 < \mathcal{R}[s]$ in which $\mathcal{R}$ means "real part of." This function is analytic at every point in this half-plane.

The region of definition of the function can be extended to include the entire plane by assuming that the function maintains the form $1/s$ throughout this extended region. By this reasoning it becomes permissible to examine the function $1/s$ at points other than where $0 < \mathcal{R}[s]$.

It will be seen that in this extended region the function $1/s$ is analytic everywhere except at the origin, and there it has a first-order pole. Thus the origin, which in this case is the point of greatest interest in the entire plane, is brought within the range of definition of the function.

B. $\mathfrak{L}$ TRANSFORMATION OF SIMPLE FUNCTIONS

The $\mathfrak{L}$ transformation will be used in this book for two purposes: (1) to transform functions of a real variable into functions of a complex variable, and (2) to transform operations such as differentiation, integration, translation, and differencing in the real domain into simpler operations in the complex domain. This second use plays an important role in the solution of differential equations and difference equations, and much that follows in subsequent chapters will pertain to this application. Our immediate concern, however, will be with the first-named use, or more specifically, with the transformation of certain simple functions of the real variable which will arise frequently in the work that follows. The results found here will be useful throughout the rest of the book and should be well known by anyone using the $\mathfrak{L}$ transformation in the solution of equations.

6. Certain Properties of the $\mathfrak{L}$-Transformation Integral

The $\mathfrak{L}$ transformation involves an improper integral in that its upper limit of integration is infinite. Moreover, the integral can be improper as a result of the behavior of its integrand at any point in the range of integration, and in particular at the lower limit of this range. Whenever this transformation is used, therefore, it will be understood that the integral is defined by a limit process. That is,

$$\int_0^\infty f(t)e^{-st}dt \triangleq \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_\epsilon^T f(t)e^{-st}dt. \qquad [14]$$

This limit process will be shown explicitly in the first example that follows, but thereafter it will not be indicated specifically as a step in the solution unless some ambiguity might arise through its omission.

The integral in the $\mathfrak{L}$ transformation, equation 14, will be taken in the Lebesgue sense [Bu 1, pp. 366–367; Ho 1, To 1]. This will make possible certain freedom in the use later of the $\mathfrak{L}$ transformation that would not be permissible otherwise. If the integral were taken in the Riemann sense the integral of a *finite* sum of functions would be equal to the sum of the integrals of the separate functions, but the same would not always be true if the integrand were an *infinite* sum of functions. Stating this in other words, it would not be permissible to replace the sum of an infinite set of integrals with a single integral. Yet it is essential to the reasoning in certain cases to follow (e.g., Sec. 6, Chapter 8) that the limit processes of $\mathfrak{L}$ transformation and infinite summation be commutative. The $\mathfrak{L}$ integral will possess this property if it is defined

in the Lebesgue sense. In treating the functions arising in most physical problems, however, the results provided by improper Lebesgue integrals will be the same as those provided by improper Riemann integrals.

7. Unity; Unit Step Function

Unity may be considered to be a function $f(t)$ that is a constant 1 for all values of t. The $\mathfrak{L}[1]$, if it exists, will be given by the integral

$$\int_0^\infty 1 \cdot e^{-st}dt = \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_\epsilon^T e^{-st}dt = \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \frac{1}{s}(e^{-s\epsilon} - e^{-sT}). \qquad [15]$$

Since the limit does exist and equals s^{-1} provided $0 < \sigma$, the $\mathfrak{L}[1] = s^{-1}$ for $0 < \sigma$.

Since the unit step function $u(t)$ is defined as unity for $0 < t$ and the integration in the $\mathfrak{L}$ transformation is from 0 to ∞, $\mathfrak{L}[u(t)] = \mathfrak{L}[1] = s^{-1}$, $0 < \sigma$. Thus the transform of both unity and $u(t)$ is the simple rational fraction s^{-1}.

Although the transformation integral defines the transform s^{-1} only in the half-plane in which $0 < \sigma$, the boundary of this region can be extended to include the entire finite plane by insisting that the form s^{-1} shall be preserved throughout this extended region. The function s^{-1} is analytic in the finite s-plane except at the origin where it has a pole of first order.

The unit step function $u(t)$ when introduced in Sec. 5, Chapter 3, was left undefined at $t = 0$. The step occurs at the origin. Even if the function is defined so as to have an explicit value at the origin, this value will not influence its $\mathfrak{L}$ transform. For example, let the following five functions be defined as stated:

Function	$t < 0$	$t = 0$	$0 < t$
$u(t)$	0	—	1
$u_a(t)$	0	$\frac{1}{2}$	1
$u_b(t)$	0	$\frac{3}{4}$	1
$u_c(t)$	0	1	1
$u_d(t)$	0	2	1

These five functions differ only in their definition at the origin. In accordance with the concept of limited equality expressed by the symbol $(=)$,

$$u_a(t), \quad u_b(t), \quad u_c(t), \text{ and } u_d(t) \ (=) \ u(t), \qquad [16]$$

and the $\mathfrak{L}$ transform of each is s^{-1} with $0 < \sigma$. This fact will be significant later when considering the $\mathfrak{L}^{-1}$ transformation in Sec. 12.

8. Exponential Functions

With α a positive real number, $\mathfrak{L}[e^{-\alpha t}]$ is

$$\int_0^\infty e^{-\alpha t}e^{-st}dt = \int_0^\infty e^{-(s+\alpha)t}dt = \frac{1}{s+\alpha}, \qquad -\alpha < \sigma. \qquad [17]$$

Thus the transform of a decreasing exponential $e^{-\alpha t}$ is an algebraic rational fraction $(s + \alpha)^{-1}$. An extended function (see Sec. 5) of the form $(s + \alpha)^{-1}$ is analytic in the finite s-plane except at $-\alpha$ where there is a first-order pole.

It follows directly that for an increasing exponential $e^{\alpha t}$ the $\mathfrak{L}$ transform is $(s - \alpha)^{-1}$, $\alpha < \sigma$. The pole of the extended function is now at $+\alpha$, and the region of convergence for the direct transformation has been reduced slightly to $\alpha < \sigma$.

9. Sinusoidal and Damped Oscillations

With β a positive real number, $\mathfrak{L}$ [sin βt] is

$$\int_0^\infty \sin \beta t \cdot e^{-st}dt = \frac{1}{2j}\int_0^\infty (e^{j\beta t} - e^{-j\beta t})e^{-st}dt$$

$$= \frac{1}{2j}\int_0^\infty [e^{-(s-j\beta)t} - e^{-(s+j\beta)t}]dt$$

$$\frac{1}{2j}\left(\frac{1}{s-j\beta} - \frac{\quad}{s+j\beta}\right) = \frac{\beta}{s^2 +} \qquad 0 < \sigma. \qquad [18]$$

An extended function of the form $\beta/(s^2 + \beta^2)$ is analytic in the finite s-plane except at the two points $\pm j\beta$ where there are first-order poles. Thus the transform of a sine function with zero initial phase angle is an algebraic rational fraction with conjugate first-order poles on the axis of imaginaries.

By steps similar to those in equation 18, $\mathfrak{L}[\cos \beta t]$ is

$$\int_0^\infty \cos \beta t \cdot e^{-st}dt = \frac{\quad}{s^2 + \beta^2}, \qquad 0 < \sigma. \qquad [19]$$

Consider now the sinusoidal function $A \cos (\beta t + \psi)$ of amplitude A and initial phase angle ψ. Since

$$A \cos (\beta t + \psi) \equiv A \cos \psi \cos \beta t - A \sin \psi \sin \beta t,$$

$\mathfrak{L}[A \cos (\beta t + \psi)]$ can be found directly by use of results 18 and 19.

$$\int_0^\infty A\cos(\beta t+\psi)\cdot e^{-st}dt = A\cos\psi\int_0^\infty \cos\beta t\cdot e^{-st}dt$$

$$-A\sin\psi\int_0^\infty \sin\beta t\cdot e^{-st}dt$$

$$=\frac{(A\cos\psi)s}{s^2+\beta^2}-\frac{A\beta\sin\psi}{s^2+\beta^2}$$

$$=\frac{a_1 s+a_0}{s^2+\beta^2},\qquad 0<\sigma, \tag{20}$$

in which $a_0 \triangleq -A\beta\sin\psi$ and $a_1 \triangleq A\cos\psi$.

Finally, a damped oscillation $e^{-\alpha t}\sin\beta t$ will be considered, α and β being positive real numbers. $\mathfrak{L}[e^{-\alpha t}\sin\beta t]$ is

$$\int_0^\infty e^{-\alpha t}\sin\beta t\cdot e^{-st}dt = \frac{1}{2j}\int_0^\infty (e^{j\beta t}-e^{-j\beta t})e^{-(s+\alpha)t}dt$$

$$=\frac{1}{2j}\int_0^\infty [e^{-(s+\alpha-j\beta)t}-e^{-(s+\alpha+j\beta)t}]dt$$

$$=\frac{1}{2j}\left(\frac{1}{s+\alpha-j\beta}-\frac{1}{s+\alpha+j\beta}\right)$$

$$=\frac{\beta}{(s+\alpha)^2+\beta^2},\qquad -\alpha<\sigma. \tag{21}$$

An extended rational function of the form of the fraction $\beta/[(s+\alpha)^2+\beta^2]$ is analytic except at the two conjugate points $-\alpha \pm j\beta$ at which there are first-order poles.

In summary it will be seen that the exponential function with real exponent treated in Sec. 8 has in this section been generalized to include exponents that are complex.

10. Positive Powers of t

$\mathfrak{L}[t]$ is given by the integral $\int_0^\infty te^{-st}dt$. Integrate by parts, letting $u=t$, and $dv=e^{-st}dt$ in the equation $\int u\,dv = uv-\int v\,du$. Then $du=dt$ and $v=\int e^{-st}dt=\dfrac{-e^{-st}}{s}$; hence

$$\int_0^\infty te^{-st}dt = \frac{-te^{-st}}{s}\Bigg|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt = 0+\frac{1}{s^2}=\frac{1}{s^2},\qquad 0<\sigma. \tag{22}$$

The function s^{-2} is an algebraic rational fraction, and when extended is analytic in the finite plane except at the origin where there is a pole of second order.

$\mathfrak{L}[t^n]$, with n a positive integer, can be found by integrating by parts n times, starting with the integral $\int_0^\infty t^n e^{-st}dt$. Let $u = t^n$ and $dv = e^{-st}dt$; then $du = nt^{n-1}dt$ and $v = \int e^{-st}dt = \frac{-e^{-st}}{s}$; hence

$$\int_0^\infty t^n e^{-st}dt = -\frac{t^n e^{-st}}{s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt, \qquad 0 < \sigma. \tag{23}$$

Integrating this result by parts

$$\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt = \frac{n}{s}\left(\frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}dt\right), \qquad 0 < \sigma.$$

This procedure is continued until there is obtained finally

$$\int_0^\infty t^n e^{-st}dt = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{s^n}\int_0^\infty t^0 e^{-st}dt$$

$$= \frac{n!}{s^{n+1}}, \qquad 0 < \sigma. \tag{24}$$

Thus the transform of the nth positive power of t is a rational algebraic function of fractional form. When extended it has a pole at the origin of multiplicity $n + 1$.

11. Product of Positive Powers of t and Exponential Function

$\mathfrak{L}[te^{-\alpha t}]$ can be found quickly by making an extension of the result found in equation 22. Let α be a positive real number. Then

$$\int_0^\infty te^{-\alpha t}e^{-st}dt = \int_0^\infty te^{-(s+\alpha)t}dt = \frac{1}{(s+\alpha)^2}, \qquad -\alpha < \sigma. \tag{25}$$

The effect on the transform of t of multiplying t by the exponential $e^{-\alpha t}$ has been to shift the second-order pole from the origin to the point $-\alpha$ on the negative real axis and to move the boundary of the region of convergence of the integral to the left by the amount α.

Further, $\mathfrak{L}[t^n e^{-\alpha t}]$ is found by an extension of the result found in equation 24 to be

$$\int_0^\infty t^n e^{-\alpha t}e^{-st}dt = \int_0^\infty t^n e^{-(s+\alpha)t}dt = \frac{n!}{(s+\alpha)^{n+1}}, \qquad -\alpha < \sigma. \tag{26}$$

The effect has been to shift the $(n + 1)$-order pole corresponding to t^n to $-\alpha$ on the negative real axis. The abscissa of absolute convergence for the function has likewise moved to the left by the amount α.

12. Table of $\mathfrak{L}$-Transform Pairs

In Table 1 the results of the transformations of functions carried out in the sections immediately preceding have been listed, with certain additions, as $\mathfrak{L}$-transform pairs. This is the beginning of a more complete tabulation of transform pairs given in Appendix A, the development of which will gradually unfold as the discussion proceeds. In these tables of pairs, the functions of the complex variable with their corresponding functions of the real variable have been placed side by side, and the operators $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ and the symbols of equality have been omitted.

Although the pairs appearing in Table 1 have been derived by proceeding from the function of the real variable to the corresponding function of the complex variable, it is apparent that, having the pairs, the table can be used also in the reverse direction. In fact, this is the way in which it will in general be the more useful, and for this reason the $F(s)$ functions have been placed on the left, and all multiplying coefficients have been placed with the $f(t)$ functions on the right.

There is available in equation 17, Chapter 3, an explicit representation of the inverse transformation as a complex integral, and rules can be formulated for carrying out the indicated integration in the complex plane, but this integration represents, in comparison with the direct transformation, a process that is less well known and, in many cases, is difficult to carry out.

The difference in the difficulty of effecting the inverse as compared with the direct transformation is similar to the difference in difficulty of effecting integration and differentiation in calculus. The definition of the integral is developed as the limit of a sum, but integration in accordance with this definition is difficult to carry out and, as a consequence, most integration is treated as the inverse of differentiation. Instead of integrating a function by taking the limit of a sum, another function is sought which, when differentiated, will yield the function to be integrated. To facilitate this inverse procedure a table of integrals is formed. In recognition of this use it is customary to indicate explicitly in these tables the operation of integration. It would be equally correct to indicate instead the operation of differentiation, for in all likelihood it is by differentiation that the tables have been built up. It is their subsequent use, however, that governs their form, and integration is the operation indicated.

TABLE 1. ELEMENTARY $\mathfrak{L}$-TRANSFORM PAIRS

No	$F(s)$	σ_a	$f(t) \qquad 0 \leqq t$
1	$\dfrac{1}{s}$	0	1, or $u(t)$
2	$\dfrac{1}{s+\alpha}$	$-\alpha$	$e^{-\alpha t}$
3	$\dfrac{1}{s^2+\beta^2}$	0	$\dfrac{1}{\beta}\sin \beta t$
4	$\dfrac{s}{s^2+\beta^2}$	0	$\cos \beta t$
5	$\dfrac{a_1 s + a_0}{s^2+\beta^2}$	0	$A \cos(\beta t + \psi)$ $A \triangleq [a_1^2 + a_0^2/\beta^2]^{\frac{1}{2}}$ $\psi \triangleq \tan^{-1} \dfrac{-a_0/\beta}{a_1}$
6	$\dfrac{1}{(s+\alpha)^2+\beta^2}$	$-\alpha$	$\dfrac{1}{\beta} e^{-\alpha t} \sin \beta t$
7	$\dfrac{s+\alpha}{(s+\alpha)^2+\beta^2}$	$-\alpha$	$e^{-\alpha t} \cos \beta t$
8	$\dfrac{a_1 s + a_0}{(s+\alpha)^2+\beta^2}$	$-\alpha$	$A e^{-\alpha t} \cos(\beta t + \psi)$ $A \triangleq [a_1^2 + (a_1\alpha - a_0)^2/\beta^2]^{\frac{1}{2}}$ $\psi \triangleq \tan^{-1} \dfrac{(a_1\alpha - a_0)/\beta}{a_1}$
9	$\dfrac{1}{s^2}$	0	t
10	$\dfrac{1}{s^n}$	0	$\dfrac{1}{(n-1)!} t^{n-1}$
11	$\dfrac{1}{(s+\alpha)^2}$	$-\alpha$	$te^{-\alpha t}$
12	$\dfrac{1}{(s+\alpha)^n}$	$-\alpha$	$\dfrac{1}{(n-1)!} t^{n-1} e^{-\alpha t}$
13	$\dfrac{1}{s} e^{-as}$	0	$u(t-a)$
14	$\dfrac{1}{s}(e^{-as} - e^{-bs})$	$-\infty$	$u(t-a) - u(t-b) \qquad a < b$

Even though the direct transformation is usually much easier to carry out than the inverse, it can still happen that the direct transformation of certain functions that satisfy the criteria of $\mathfrak{L}$ transformability will be difficult if the results are wanted in closed form. As may be expected, if the function to be transformed is the free solution of a linear i-d equation with constant coefficients, its transformation is relatively easy. This transformation can be carried out by direct integration as in the examples above and, in more complicated cases, with the assistance of the theorems of Chapter 8. An alternative method is to (1) treat the function as the solution of a differential equation, forming this equation by the well-known method of eliminating the constants through differentiation, and (2) find the transform of the solution of this equation by the methods developed in Chapter 5. The solution transform found in this way will be the transform of the function. If, on the other hand, the function to be transformed is the solution of a linear i-d equation with variable coefficients or — even worse — of a nonlinear i-d equation, difficulty in computing the direct transform may be expected with any of the methods.

In the table of transform pairs presented here, neither the operation of direct transformation nor the operation of inverse transformation is indicated, as the table can be useful in either operation. It is understood that the following two equations can be formulated for each pair:

$$\begin{aligned} F(s) &= \mathfrak{L}[f(t)], \\ \mathfrak{L}^{-1}[F(s)] &(=) f(t), \qquad 0 \leq t. \end{aligned} \tag{27}$$

It has no doubt been gathered from the discussion in Sec. 7 that the direct transformation is unique and always takes the symbol of equality, whereas the inverse transformation requires in general the symbol $(=)$ of restricted equality meaning "equals almost everywhere." Two functions $f_1(t)$ and $f_2(t)$ may have equal continuous portions but be unequal at points of discontinuity. They will both have the same $\mathfrak{L}$ transform $F(s)$, but this $F(s)$ may have either $f_1(t)$ or $f_2(t)$ for its $\mathfrak{L}^{-1}$ transform. The symbol $(=)$ indicates this possibility.

C. BASIC $\mathfrak{L}$-TRANSFORMATION THEOREMS

After the preceding interlude on the theory of functions of a complex variable, and the use of the $\mathfrak{L}$ operator to transform simple functions, it will now be easier to understand several theorems on the Laplace transformation. These will make more precise the ideas introduced in Chapter 3.

DEFINITION. *A real*[1] *function* f(t) *which is defined and single valued almost everywhere for* $0 \leqq t$, *with* t *a real variable, and is such that the improper Lebesgue integral* [Ho 1, To 1]

$$\int_0^\infty |f(t)|e^{-\sigma t}dt \triangleq \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_\epsilon^T |f(t)|e^{-\sigma t}dt < \infty \tag{28}$$

for some real number σ, *will be called* $\mathfrak{L}$ *transformable.*

With any specified $f(t)$, the (greatest) lower bound of all the real numbers which satisfy condition 28 is called the *abscissa of absolute convergence* corresponding to that $f(t)$. It is denoted by σ_a.

13. THEOREM 1, $\mathfrak{L}$ TRANSFORMATION

If f(t) *is* $\mathfrak{L}$ *transformable, then the Laplace integral (improper Lebesgue integral),*

$$\int_0^\infty f(t)e^{-st}dt \triangleq \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_\epsilon^T f(t)e^{-st}dt, \tag{29}$$

with s *a complex variable* $\sigma + j\omega$, *converges absolutely for* $\sigma_a < \sigma$ *to a function* F(s) *which is analytic in the half-plane* $\sigma_a < \sigma$ [HA 5].

As mentioned in Chapter 3 (p. 105), this functional transformation will be written in abbreviated notation as

$$\mathfrak{L}[f(t)] = F(s), \qquad \sigma_a < \sigma. \tag{30}$$

14. THEOREM 2, $\mathfrak{L}^{-1}$ TRANSFORMATION

The next definition follows immediately from the notion of the inverse of a functional transformation and from Theorem 1.

DEFINITION. *The inverse Laplace transformation* $\mathfrak{L}^{-1}$ *is defined implicitly here by the relation*

$$\mathfrak{L}^{-1}\{\mathfrak{L}[f(t)]\} \;(=)\; f(t), \qquad 0 \leqq t. \tag{31}$$

This can be written: If $F(s) = \mathfrak{L}[f(t)]$, then for $0 \leqq t$, $f(t)$ (=) $\mathfrak{L}^{-1}[F(s)]$.

The $\mathfrak{L}^{-1}$ operation can be given an explicit representation in terms of known mathematical operations, as shown by the following theorem:

[1] Complex functions of a real variable can be treated by resolving them into real and imaginary parts which are real.

If F(s) *is the* $\mathfrak{L}$ *transform of a function* f(t), *then*

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{ts}ds \;(=)\; f(t), \qquad 0 \leqq t, \tag{32}$$

in which $\sigma_a < c$ [ME 1, APPEN C].

Thus an explicit representation of $\mathfrak{L}^{-1}$ [ ] is

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} [\quad] e^{ts}ds, \qquad \sigma_a < c. \tag{33}$$

" Explicit representation " means here a functional transformation that can be effected directly. In contrast to this the above definition 31 of $\mathfrak{L}^{-1}$ defines $\mathfrak{L}^{-1}$ implicitly. To effect the $\mathfrak{L}^{-1}$ transformation using only this implicit definition it is necessary (1) to form a table of transform pairs by using the $\mathfrak{L}$ transformation and (2) to use the proper pair in the inverse order from that in which it was obtained.

For the present the explicit representation which is shown in expression 33 and was mentioned in Sec. 6, Chapter 3, will not be needed. Other explicit representations are known [WI 2, BO 0].

15. THEOREM 3, UNICITY OF $\mathfrak{L}$ TRANSFORM

Theorem 3 states a property that follows from the defining integral 29.

If f(t) *is* $\mathfrak{L}$ *transformable and* $\mathfrak{L}[f(t)] = F(s)$, $\sigma_a < \sigma$, *then its* $\mathfrak{L}$ *transform* F(s) *is unique* [LE 2].

Example 1. Consider $f_1(t) \triangleq u(t-b)$, $0 < b$. $u(t-b)$ is $\mathfrak{L}$ transformable ($\sigma_a = 0$), and by direct evaluation of the integral 29, $\mathfrak{L}[u(t-b)] = e^{-bs}s^{-1}$, $0 < \sigma$. Here $e^{-bs}s^{-1}$ is the unique $\mathfrak{L}$ transform of $u(t-b)$.

16. THEOREM 4, LEBESGUE UNICITY OF $\mathfrak{L}^{-1}$ TRANSFORM

From the fact that a Lebesgue integral appears in the condition 28 that a function $f(t)$ be $\mathfrak{L}$ transformable, it follows that any other function which is equal almost everywhere to $f(t)$ will also be $\mathfrak{L}$ transformable. In the same way, from the fact that the defining integral 29 for the $\mathfrak{L}$ transformation is a Lebesgue integral, it follows that these other functions which are equal almost everywhere to $f(t)$ will have the same $\mathfrak{L}$ transform as $f(t)$. This leads to the fourth theorem.

If f(t) *is an* $\mathfrak{L}^{-1}$ *transform of* F(s), *then* f(t) *is Lebesgue unique, i.e., all other* $\mathfrak{L}^{-1}$ *transforms of* F(s) *are equal to* f(t) *almost everywhere for* $0 \leqq t$.

In symbols this can be written: $\mathfrak{L}^{-1}[F(s)] \;(=)\; f(t), \qquad 0 \leqq t.$

Example 1. Consider the two step functions $u_a(t)$ and $u_c(t)$ defined on page 115. Recalling Ex. 1 in Sec. 15, it is permissible to write $\mathfrak{L}^{-1}[e^{-bs}s^{-1}]$ (=) $u_a(t-b)$. But it is also true that $\mathfrak{L}^{-1}[e^{-bs}s^{-1}]$ (=) $u_c(t-b)$. Thus $u_a(t-b)$ is a Lebesgue-unique $\mathfrak{L}^{-1}$ transform of $e^{-bs}s^{-1}$.

PROBLEMS

4·1. (*a*) If $F_1(s) \triangleq (s^2+1)^{-1}$, are there points in the s-plane at which $F_1(s)$ is not analytic? Explain.

(*b*) If $[F_2(s)]^2 \triangleq (s^2+2s+1)^{-1}$, is $F_2(s)$ analytic at any point in the s-plane? Explain.

4·2 Show that the $\mathfrak{L}$ transform of a unit step function which has been translated to the right by the amount a is $s^{-1}e^{-as}$.

4·3. (*a*) Show that the rectangular pulse of height unity and located between points a and b, with $a < b$, can be represented by each of the following expressions:

1. $u(t-a) - u(t-b)$
2. $u(b-t) - u(a-t)$
3. $u(t-a) \cdot u(b-t)$.

(*b*) Show that the $\mathfrak{L}$ transform of the rectangular pulse described in part (*a*) is $s^{-1}(e^{-as} - e^{-bs})$.

4·4. For each of the functions given below answer question (*a*) and if your answer is "yes" answer questions (*b*), (*c*), and (*d*).

(*a*) Is the function $\mathfrak{L}$ transformable?

(*b*) What is its σ_a?

(*c*) What is its $\mathfrak{L}$ transform?

(*d*) At what points in the finite portion of the complex plane does its $\mathfrak{L}$ transform, when extended, have poles and what is the order of each pole?

The functions to be considered are:

1. $\dfrac{1}{\alpha-\beta}(\alpha e^{-\alpha t} - \beta e^{-\beta t})$.
2. $\sin(\beta t + \psi)$.
3. $\epsilon^{-\alpha t}\sin(\beta t + \psi)$.
4. $te^{-\alpha t}\sin\beta t$.
5. b^t, $b \neq 0$ or 1.
6. t^t.
7. $\begin{cases} 0 & t < 0 \\ t/a & 0 < t < a \\ 1 & a < t. \end{cases}$
8. A single triangular-shaped pulse starting at the origin. Its base is 2, its maximum height is unity, and it is isosceles.
9. $1 + \sum_{k=1}^{\infty}(-1)^k u(t-k)$, k is positive integer.
10. $t - \sum_{k=1}^{\infty} u(t-k)$, k is positive integer.

NOTE: α, β, ψ, a, and b are non-negative real numbers. For parts 9 and 10, express the answer to (*c*) in closed form, i.e., sum the series, before answering (*d*).

4-5. Show that the following constitutes an $\mathfrak{L}$-transform pair:

$$\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{s^2/4a}\ \operatorname{cerf}\frac{s}{2\sqrt{a}},$$

in which

$$\operatorname{cerf} y \triangleq 1 - \operatorname{erf} y \triangleq 1 - \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}dx.$$

CHAPTER V

THE $\mathfrak{L}$ TRANSFORMATION OF INTEGRODIFFERENTIAL EQUATIONS IN ONE INDEPENDENT VARIABLE

Having discussed the formulation of one-dimensional linear integrodifferential equations with constant coefficients in Chapter 2 and having introduced the Laplace transformation in Chapters 3 and 4, the application of this transformation to equations of this type can now be presented. It is applied first to a single equation and later to a set of equations. The immediate objective is to illustrate the first step in the $\mathfrak{L}$-transformation method of solving such equations [APPEN C, Do 15, ST 1]. In this first step the i-d equations are converted into algebraic equations, and in the process provision is made for the initial conditions. The second step is the algebraic solution of these equations. This yields functions of the complex variable, the inverse transformation of which constitutes the third step in the solution. This third step is presented in Chapters 6 and 7.

An i-d equation in one dimension, as seen in Chapter 2, contains an unknown function, integrals and ordinary derivatives of this function, constants, and the known driving function. In addition, there must be a supplementary statement of initial or boundary conditions. The functions all have for their argument a real variable, and since they arise from physical problems are usually $\mathfrak{L}$ transformable into functions of a complex variable. Since the coefficients in the equation are constants, it will be seen shortly that one direct transformation applied to the entire equation eliminates all the integrals and derivatives and makes evident all the necessary initial conditions.

Two notable results of applying the $\mathfrak{L}$ transformation are: (1) an i-d equation is replaced by an algebraic equation which is solvable by algebraic methods, and (2) this simpler equation contains all the information essential to the complete solution of the particular problem that was set.

The $\mathfrak{L}$ transformation of the i-d equation is carried out by multiplying both of its members by the exponential kernel e^{-st} and integrating with respect to t with the limits 0 and ∞. As mentioned in Chapter 3, page 101, the origin for the variable t is so chosen that only positive values of t need to be considered.

In Chapter 4 the $\mathfrak{L}$ transformation of *functions* that satisfy certain conditions was discussed. To transform i-d equations the three following theorems for the $\mathfrak{L}$ transformation of *operations* will be needed. They deal with the transformation of (1) a sum of functions, (2) the derivative of a function, and (3) the integral of a function.

1. Theorem 5, Linearity

If the functions f(t), f_1(t), *and* f_2(t) *are $\mathfrak{L}$ transformable and have $\mathfrak{L}$ transforms* F(s), F_1(s), *and* F_2(s), *respectively, and* a *is a constant or a variable which is independent of* t *and* s, *then*

$$\mathfrak{L}[af(t)] = aF(s), \tag{1}$$

and

$$\mathfrak{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s). \tag{2}$$

If the functions F(s), F_1(s), *and* F_2(s) *are $\mathfrak{L}$ transforms of functions* f(t), f_1(t), *and* f_2(t), *respectively, and* a *is a constant or a variable which is independent of* s *and* t, *then*

$$\mathfrak{L}^{-1}[aF(s)] \;(=)\; af(t), \quad 0 \leqq t, \tag{1'}$$

and

$$\mathfrak{L}^{-1}[F_1(s) \pm F_2(s)] \;(=)\; f_1(t) \pm f_2(t), \quad 0 \leqq t. \tag{2'}$$

This theorem defines the linear character of the $\mathfrak{L}$ transformation and its inverse. The proof follows directly from the linear property of the integral defining the $\mathfrak{L}$ transformation and the definition of the $\mathfrak{L}^{-1}$ transformation.

2. Theorem 6, Real Differentiation

If the function f(t) *and its derivative* $\frac{\mathrm{df(t)}}{\mathrm{dt}}$ *are $\mathfrak{L}$ transformable, and if* f(t) *has the $\mathfrak{L}$ transform* F(s), *then*

$$\mathfrak{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0+). \tag{3}$$

In discussing this theorem and others to follow, it will be convenient to call the domain of the real variable the *real domain* and the domain of the complex variable the *complex domain*.

Theorem 6 states that differentiation with respect to the variable in the real domain goes over in the complex domain into multiplication by the complex variable to within an additive constant $f(0+)$.

Since functions with finite discontinuities arise naturally in the problems to be solved here, it is necessary to distinguish between a derivative

defined as the limit of a difference quotient in which the increment approaches zero from the right (called a right-hand derivative) and one in which the increment approaches zero from the left (called a left-hand derivative). If both exist and are equal, the result is called the derivative. At points of discontinuity only right-hand derivatives will be of concern unless the contrary is stated, and the derivatives will be represented by the usual derivative symbols $\dfrac{df}{dt}$ and $f'(t)$.

The statement of Theorem 6 anticipates the possibility that the function may have a step at the origin. The additive term $f(0+)$ is the value of the function as the origin is approached from the positive or right side. For convenience, the $+$ sign appended to the zero will usually be omitted hereafter, but its presence will be implied in the use of the theorem.

It should be noted that the theorem requires that the derivative, as well as the function, be $\mathfrak{L}$ transformable.

The demonstration of this theorem proceeds from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0 f(t)e^{-st}dt = F(s),$$

by an integration by parts. Let $u = f(t)$ and $dv = e^{-st}dt$ in

$$\int udv = uv - \int vdu.$$

This gives

$$\int_0^\infty f(t)e^{-st}dt = -\frac{1}{s}f(t)e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}dt$$

$$= \frac{f(0+)}{s} + \frac{1}{s}\int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}dt. \qquad [4]$$

The presence of $0+$ is understood when the definition of the $\mathfrak{L}$ transformation is recalled. The lower limit ϵ of the defining integral approaches 0 from the right. Rearrangement of terms and multiplication by s gives

$$\int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}dt = sF(s) - f(0+), \qquad [5]$$

or

$$\mathfrak{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0+),$$

as stated in the theorem.

Extension of this real-differentiation theorem to cover higher-order derivatives, when such are $\mathfrak{L}$ transformable, can be made readily by repeated applications of the process shown in equation 5. Thus

$$\begin{aligned}\mathfrak{L}[f'(t)] &= sF(s) - f(0),\\ \mathfrak{L}[f''(t)] &= s[sF(s) - f(0)] - f'(0)\\ &= s^2F(s) - f(0)s - f'(0),\end{aligned}$$

$$\mathfrak{L}[f^{(n)}(t)] = s^nF(s) - \sum_{k=1}^{n} f^{(k-1)}(0)s^{n-k}, \tag{6}$$

in which $f^{(k)}(t) \triangleq \dfrac{d^kf(t)}{dt^k}$ and $f^{(0)}(t) \triangleq f(t)$.

3. THEOREM 7, REAL INTEGRATION

If the function f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *its integral*

$$f^{(-1)}(t) = \int f(t)dt = \int_0^t f(t)dt + f^{(-1)}(0+) \tag{7}$$

is likewise $\mathfrak{L}$ *transformable, and*

$$\mathfrak{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}. \tag{8}$$

This theorem states that integration with respect to the variable in the real domain goes over in the complex domain into division by the variable to within an additive constant of integration divided by the variable, i.e., $\dfrac{f^{(-1)}(0+)}{s}$.

That the integral of a function is transformable if the function itself is transformable follows from consideration of the requirements for $\mathfrak{L}$ transformability. If $f(t)$ is defined and single valued almost everywhere for $0 \leqq t$, its integral in this region will likewise be defined and single valued. If a real number σ can be found such that the product function $f(t)e^{-\sigma t}$ is absolutely integrable for $0 \leqq t$, then from the nature of an integral it will likewise be possible to find a real number σ_1 such that the product function $e^{-\sigma_1 t}\int f(t)dt$ is absolutely integrable in the range $0 \leqq t$. Thus by fulfilling these three conditions the integral of $f(t)$ is $\mathfrak{L}$ transformable.

Here, as in Theorem 6, provision is made for treatment of functions less well behaved than the usual continuous type. If $f^{(-1)}(t)$ has a step at the origin, the constant of integration $f^{(-1)}(0+)$ is the value of $f^{(-1)}(t)$ as the origin is approached from the positive or right side. For brevity, the + sign will usually be omitted hereafter.

The demonstration of this theorem proceeds from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(t)e^{-st}dt = F(s),$$

by an integration by parts. Reversing here the choice of substitution that was made in proving Theorem 6, let $u = e^{-st}$ and $dv = f(t)dt$ in

$$\int udv = uv - \int vdu.$$

This gives

$$\int_0^\infty f(t)e^{-st}dt = e^{-st}\int f(t)dt\Big|_0^\infty + s\int_0^\infty \left[\int f(t)dt\right]e^{-st}dt$$

$$= -f^{(-1)}(0+) + s\int_0^\infty \left[\int f(t)dt\right]e^{-st}dt. \qquad [9]$$

Upon rearrangement of terms and division by s, equation 9 becomes

$$\int_0^\infty \left[\int f(t)dt\right]e^{-st}dt = \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}, \qquad [10]$$

or

$$\mathfrak{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}$$

as stated in the theorem.

Extension of this real-integration theorem to cover higher-order integrals can be made readily by its repeated application. Let $f^{(-k)}(t) \triangleq \int \cdots \int f(t)(dt)^k$, and $f^{(-0)}(t) \triangleq f(t)$, then

$$\mathfrak{L}[f^{(-1)}(t)] = \frac{F(s)}{s} + \frac{f^{(-1)}(0)}{s},$$

$$\mathfrak{L}[f^{(-2)}(t)] = \frac{F(s)}{s^2} + \frac{f^{(-1)}(0)}{s^2} + \frac{f^{(-2)}(0)}{s},$$

$$\mathfrak{L}[f^{(-n)}(t)] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{f^{(-k)}(0)}{s^{n-k+1}}. \qquad [11]$$

4. SECOND-ORDER DIFFERENTIAL EQUATION

Having discussed the $\mathfrak{L}$ transformation of certain common functions and operations, this transformation will now be applied to the differential equation

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t), \qquad y \triangleq y(t), \tag{12}$$

in which A, B, and C are known constants. The unknown $y(t)$ will be called the *reponse function*, and the known $f(t)$ will be called the *driving function*. The initial values of the unknown and its first derivative are $y(0)$ and $y'(0)$.

This being the first application of the method, the steps will be carried out in more detail than will be necessary in subsequent applications.

Applying the $\mathfrak{L}$ transformation to both members of equation 12,

$$\mathfrak{L}\left[A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy\right] = \mathfrak{L}\,[f(t)]. \tag{13}$$

The driving function $f(t)$ is assumed to be $\mathfrak{L}$ transformable and to have the $\mathfrak{L}$ transform $F(s)$. $F(s)$ will be called the *driving transform.*

Since the response function $y(t)$ and its first and second derivatives are unknown, the question may well arise: How does one know whether or not the left member is $\mathfrak{L}$ transformable? The answer to this question is not obtainable at this stage in the solution, but it is possible to proceed on the assumption that equation 12 has a solution $y(t)$. Furthermore, assume that this solution is $\mathfrak{L}$ transformable, and let

$$\mathfrak{L}[y(t)] \triangleq Y(s). \tag{14}$$

Here, as with most other methods of solution, the proof that $y(t)$, when found, is a solution must be that it satisfies not only the original differential equation but also the prescribed initial conditions. $Y(s)$ will be called the *response transform.*

Assuming that each of the derivatives $y'(t)$ and $y''(t)$ is $\mathfrak{L}$ transformable, application of Theorem 6 provides that

$$\left.\begin{aligned} \mathfrak{L}[y'(t)] &= sY(s) - y(0), \\ \mathfrak{L}[y''(t)] &= s^2Y(s) - y(0)s - y'(0). \end{aligned}\right\} \tag{15}$$

This discloses the way in which the initial conditions $y(0)$ and $y'(0)$ are incorporated in the solution during the process of transformation.

Returning to equation 13 and applying Theorem 5, the operation of transformation is distributed over the three terms of the left member.

The result is

$$A\mathfrak{L}\left[\frac{d^2y}{dt^2}\right] + B\mathfrak{L}\left[\frac{dy}{dt}\right] + C\mathfrak{L}[y] = \mathfrak{L}[f(t)]$$

and upon substitution from equations 14 and 15 becomes

$$A[s^2Y(s) - y(0)s - y'(0)] + B[sY(s) - y(0)] + CY(s) = F(s), \qquad [16]$$

which can be rewritten as

$$(As^2 + Bs + C)Y(s) = F(s) + y(0)(As + B) + y'(0)A. \qquad [17]$$

An algebraic equation, such as 16 or 17, obtained from the $\mathfrak{L}$ transformation of a differential (or i-d, or difference) equation will be called a *transform equation.*

The polynomial coefficient of $Y(s)$ — in this case $As^2 + Bs + C$ — will be called the *characteristic function,* since it completely characterizes the physical system described by the differential equation. It contains the constants and the information on the interconnection or geometry of the system. The equation formed by setting it equal to zero is called the *characteristic equation* of the system.

Solving equation 17 algebraically,

$$Y(s) = \frac{1}{As^2 + Bs + C}[F(s) + y(0)(As + B) + y'(0)A]. \qquad [18]$$

This algebraic solution has a form which will be found typical of all transform solutions. It is

response transform = (system function) · (excitation function).

The *system function* in this example is the reciprocal of the characteristic function, but in general it will be a fraction of which the characteristic function is the denominator. It incorporates in one function all the essential knowledge regarding the physical system.

The *excitation function* includes the driving transform and the initial conditions, the latter in the form of an *initial excitation function.* It contains all the essential specifications of the excitations applied to the system.

$Y(s)$ was taken as the transform of $y(t)$. When the form of the driving function $f(t)$ is specified, the algebraic form of $Y(s)$ can be determined. Applying the $\mathfrak{L}^{-1}$ transformation to both members of equation 18,

$$\mathfrak{L}^{-1}[Y(s)] = \mathfrak{L}^{-1}\left[\frac{F(s) + y(0)(As + B) + y'(0)A}{As^2 + Bs + C}\right]. \qquad [19]$$

Carrying out the indicated operation on the left member, there results for $0 \leqq t$,

$$y(t) \ (=) \ \mathfrak{L}^{-1}\left[\frac{F(s) + y(0)(As + B) + y'(0)A}{As^2 + Bs + C}\right]. \qquad [20]$$

If $Y(s)$ were an algebraic function of the form of any one of the various transforms listed in Table 1, Chapter 4, its inverse could be written immediately by reference to that table. But since $Y(s)$ is a more complicated function than any listed there, such a direct method of determining its inverse transform fails. A difficulty of this nature arises in ordinary integration when the integral to be evaluated does not come within the range of the table of integrals that is available. The method of surmounting it is to resolve the function to be integrated into a sum of simpler components the integrals of each of which may appear separately in the table. Unless one resorts to a procedure like this the table of integrals must of necessity be long and unwieldy. The present difficulty can be surmounted in the same manner, namely, by resolving the transform $Y(s)$ into a sum of simpler component transforms each of which comes within the scope of Table 1, Chapter 4.

The final step indicated in equation 20 will be postponed, however, until certain general principles of the $\mathfrak{L}^{-1}$ transformation applicable to all rational algebraic transforms have been developed in the following chapter. The remainder of the present chapter will be used for a number of examples, the solution of each being carried as far as the indication of the final inverse transformation.

5. First-Order Integrodifferential Equation

As a second example the $\mathfrak{L}$ transformation will be applied to the first-order integrodifferential equation

$$A\frac{dy}{dt} + By + C\int y dt = f(t), \qquad y \triangleq y(t), \qquad [21]$$

Let the initial values of the unknown and its first integral be $y(0)$ and $y^{(-1)}(0)$.

In the previous section the procedure to be applied in the solution of an ordinary differential equation by the $\mathfrak{L}$-transformation method was presented in great detail. The procedure here will be abbreviated to only the major steps.

Assume that $y(t)$, $\dfrac{dy(t)}{dt}$, and $f(t)$ are all $\mathfrak{L}$ transformable, and let

$$\mathfrak{L}[y(t)] \triangleq Y(s),$$
$$\mathfrak{L}[f(t)] \triangleq F(s).$$

Then by Theorem 6,

$$\mathfrak{L}[y'(t)] = sY(s) - y(0).$$

And by Theorem 7,

$$\mathfrak{L}[y^{(-1)}(t)] = \frac{Y(s)}{s} + \frac{y^{(-1)}(0)}{s}.$$

With the application of Theorem 5, the transform equation is

$$A\,[sY(s) - y(0)] + BY(s) + C\left[\frac{1}{s}\,Y(s) + \frac{y^{(-1)}(0)}{s}\right] = F(s), \qquad [22]$$

which can be rewritten as

$$\left(As + B + \frac{C}{s}\right)Y(s) = F(s) + y(0)A - \frac{y^{(-1)}(0)C}{s}. \qquad [23]$$

Here the characteristic function is the polynomial $As^2 + Bs + C$. Solving equation 23 for $Y(s)$ gives

$$Y(s) = \frac{F(s) + y(0)A - y^{(-1)}(0)C\,\dfrac{1}{s}}{As + B + \dfrac{C}{s}}. \qquad [24]$$

Equation 24 can be written in the form

response transform = (system function) · (excitation function),

and contains all the essential information of the problem.

In accordance with the assumption that $Y(s)$ is the $\mathfrak{L}$ transform of $y(t)$, the response function is found from the $\mathfrak{L}^{-1}$ transformation of $Y(s)$. That is, for $0 \leq t$,

$$y(t)\ (=)\ \mathfrak{L}^{-1}\left[\frac{F(s) + y(0)A - y^{(-1)}(0)C\,\dfrac{1}{s}}{As + B + \dfrac{C}{s}}\right] \qquad [25]$$

and the solution is reduced to the final steps of expressing $Y(s)$ as a sum of terms, each recognizable with the aid of Table 1, Chapter 4, as the transform of a particular function of time.

6. Equation for One-Loop Electric Network

The circuit of Fig. 5·1 is operating in the steady state with the switch K open. This switch is suddenly closed at an instant when the sinusoidal

driving voltage is zero and has a positive derivative. At this instant, which may be called $t = 0$, there is energy stored in the inductance and in the capacitance. The $\mathfrak{L}$ transform for the circuit current $i(t)$ after this switching will now be found.

The initial energy conditions can be accounted for by specifying the initial current in the inductance and the initial voltage across the capacitance. Let the magnitude of the initial current in L be ρ, and the magnitude of the initial voltage across C be γ. Let the sense of ρ and the polarity of γ be as indicated in the diagram.

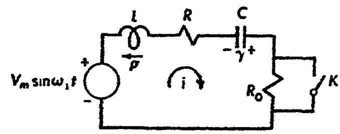


FIG. 5·1

The i-d equation for the circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = V_m \sin \omega_1 t, \qquad i \triangleq i(t). \tag{26}$$

Introducing the condenser initial condition into the equation, 26 becomes

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t idt = V_m \sin \omega_1 t + \gamma. \tag{27}$$

By pairs 3 and 1, Table 1, Chapter 4, the

$$\mathfrak{L}[V_m \sin \omega_1 t + \gamma] = \frac{V_m\omega_1}{s^2 + \omega_1^2} + \frac{\gamma}{s}. \tag{28}$$

Then letting $\mathfrak{L}[i(t)] \triangleq I(s)$, the $\mathfrak{L}$ transformation of equation 27 gives

$$L[sI(s) - i(0)] + RI(s) + \frac{1}{Cs}I(s) = \frac{V_m\omega_1}{s^2 + \omega_1^2} + \frac{\gamma}{s}. \tag{29}$$

Note that the result would have been identical if 26 instead of 27 had been the equation transformed. The real integration theorem brings in the initial condenser voltage when the indefinite integral is transformed; hence the condenser initial voltage need not be introduced into the i-d equation. It will be found convenient at first, however, to introduce it into the i-d equation — at least until a certain familiarity is acquired with the significance of the terms in the transform equation.

Upon collection of terms, equation 29 becomes

$$\left[Ls + R + \frac{1}{Cs}\right]I(s) = \frac{V_m\omega_1}{s^2 + \omega_1^2} + \ i(0)L + \frac{\gamma}{s}. \tag{30}$$

But $i(0) = -\rho$, the sign being minus since the initial current in L is opposite to the loop arrow direction. Solving equation 30 for $I(s)$,

$$I(s) = \frac{V_m\omega_1/(s^2 + \omega_1^2) - \rho L + \gamma/s}{Ls + R + 1/Cs} = \frac{V_m\omega_1 s - (\rho Ls - \gamma)(s^2 + \omega_1^2)}{(s^2 + \omega_1^2)(Ls^2 + Rs + 1/C)}. \qquad [31]$$

The $\mathfrak{L}^{-1}$ transformation of equation 31 gives, for $0 \leq t$,

$$i(t) \; (=) \; \mathfrak{L}^{-1}\left[\frac{V_m\omega_1 s - (\rho Ls - \gamma)(s^2 + \omega_1^2)}{(s^2 + \omega_1^2)(Ls^2 + Rs + 1/C)}\right], \qquad [32]$$

and the solution is reduced to the final steps of carrying out the inverse transformation.

7. Equations for Two-Loop Network

In Fig. 5·2 is shown the two-loop network for which the i-d equations were written in Sec. 5, Chapter 2. The driving voltage $v(t)$ is specified to be $V_m \cos \omega_1 t$. The magnitude of the initial voltage across C_2 is γ, with polarity as indicated; the magnitude of the initial current in L_1 is ρ, with direction as indicated. The $\mathfrak{L}$ transforms of the loop currents will now be found.

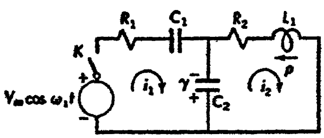


FIG. 5·2

The i-d equations for this network were given in equations 17, Chapter 2. They are

$$\left.\begin{aligned} &R_1 i_1 + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_0^t i_1 dt - \frac{1}{C_2}\int_0^t i_2 dt = V_m \cos \omega_1 t + \gamma, \\ &-\frac{1}{C_2}\int_0^t i_1 dt + L_1 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int_0^t i_2 dt = -\gamma. \end{aligned}\right\} \qquad [33]$$

By pair 4, Table 1, Chapter 4, the

$$\mathfrak{L}[V_m \cos \omega_1 t] = V_m \frac{s}{s^2 + \omega_1^2} \triangleq V(s).$$

If the $\mathfrak{L}$ transforms of $i_1(t)$ and $i_2(t)$ are denoted by $I_1(s)$ and $I_2(s)$, respectively, the $\mathfrak{L}$ transformation of equations 33 gives

$$\left.\begin{aligned} &R_1 I_1(s) + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{s} I_1(s) - \frac{1}{C_2 s} I_2(s) = V(s) + \frac{\gamma}{s}, \\ &-\frac{1}{C_2 s} I_1(s) + L_1[sI_2(s) - i_2(0)] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = -\frac{\gamma}{s} \end{aligned}\right\} \qquad [34]$$

Collecting terms in equations 34, and noting that $i_2(0) = -\rho$,

$$\left.\begin{aligned} \left[R_1 + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{s}\right]I_1(s) - \frac{1}{C_2 s} I_2(s) &= V(s) + \frac{\gamma}{s}, \\ -\frac{1}{C_2 s} I_1(s) + \left(L_1 s + R_2 + \frac{1}{C_2 s}\right) I_2(s) &= -\rho L_1 - \frac{\gamma}{s}. \end{aligned}\right\} \qquad [35]$$

At this point it is desirable to introduce a type of simplifying notation that will be useful here and also later in similar problems. Let

$$\left.\begin{aligned} z_{11}(s) &\triangleq R_1 + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{s}, \\ z_{22}(s) &\triangleq L_1 s + R_2 + \frac{1}{C_2 s}, \\ z_{12}(s) = z_{21}(s) &\triangleq -\frac{1}{C_2 s}, \\ E_1(s) &\triangleq V(s) + \frac{\gamma}{s}, \\ E_2(s) &\triangleq -\rho L_1 - \frac{\gamma}{s}. \end{aligned}\right\} \qquad [36]$$

In these abbreviations, $z_{11}(s)$ and $z_{22}(s)$ are the *self-impedance functions* of loops 1 and 2, respectively. $z_{12}(s) = z_{21}(s)$ is the *mutual-impedance function* of loops 1 and 2. It will be observed that these impedance functions can be formed directly from the corresponding a's (i.e., the self- and mutual-i-d operators of the loops) by replacing the differentiation operator $\frac{d}{dt}$ with s and the integration operator $\int_0^t [\ \ ]\,dt$ with s^{-1}. Better still, they can be written directly by inspection of the connection diagram without writing the i-d equations.

The abbreviations $E_1(s)$ and $E_2(s)$ will be called the *excitation functions* for loops 1 and 2, respectively. The excitation function for any given loop consists of the algebraic sum, taken around this loop, of the driving transforms, the initial-current-inductance products, and s^{-1} times the initial condenser voltages. The signs to be given driving transforms are the same as would be given their respective sources in writing Kirchhoff's voltage equation for the loop. In summing the initial-current-inductance products, a product resulting from an initial current in a self-inductance in the arrow direction of the loop is considered positive. Products resulting from pure mutual inductance will

be discussed in Sec. 8. In summing initial condenser voltages, a voltage rise in the arrow direction is considered positive.

Using the abbreviations of 36, equations 35 become

$$\begin{aligned} z_{11}(s)I_1(s) + z_{12}(s)I_2(s) &= E_1(s), \\ z_{21}(s)I_1(s) + z_{22}(s)I_2(s) &= E_2(s). \end{aligned} \qquad [37]$$

The transform equations 35 and 37 suggest a connection diagram which applies equally well to both transient and steady-state analysis. Instead of the usual L,R,S, and source elements, its elements symbolize inductive-reactance[1] functions, resistances, capacitive-reactance[1] functions, and loop-excitation functions. The unknowns symbolized are the loop-current transforms. Such a diagram, which will be called a *transform diagram*, is given for this example in Fig. 5·3. Note that loop 2 has an excitation function $E_2(s)$ although there was no impressed voltage in this loop. It is apparent that such a diagram can be constructed directly from the original problem statement and that solution for steady-state and transient responses can begin here, omitting both the writing of the i-d equations and their subsequent transformation.

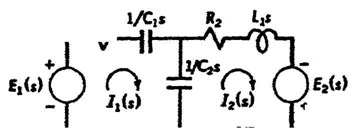


FIG. 5·3. Transform diagram for the network of Fig. 5·2.

The transform equations 37 can be solved by algebra for $I_1(s)$ and $I_2(s)$ by the substitution method. The solutions are

$$\left.\begin{aligned} I_1(s) &= \frac{z_{22}(s)}{\Delta(s)} E_1(s) + \frac{-z_{12}(s)}{\Delta(s)} E_2(s), \\ I_2(s) &= \frac{-z_{21}(s)}{\Delta(s)} E_1(s) + \frac{z_{11}(s)}{\Delta(s)} E_2(s), \end{aligned}\right\} \qquad [38]$$

with

$$\Delta(s) \triangleq z_{11}(s)z_{22}(s) - z_{12}^2(s).$$

For further simplification, let

$$\begin{aligned} Y_{11}(s) &\triangleq \frac{z_{22}(s)}{\Delta(s)} \\ Y_{22}(s) &\triangleq \frac{z_{11}(s)}{\Delta(s)} \\ Y_{21}(s) = Y_{12}(s) &\triangleq \frac{-z_{21}(s)}{\Delta(s)} = \frac{-z_{12}(s)}{\Delta(s)} \end{aligned} \qquad [39]$$

[1] The term "reactance" is used here in a generalized sense to apply to a function of a complex variable s rather than to the usual steady state function of ω.

$Y_{11}(s)$ and $Y_{22}(s)$ are the *short-circuit input-admittance functions* of the network viewed from loop 1 and loop 2, respectively. $Y_{21}(s)$ is the *short-circuit transfer-admittance function* for loops 1 and 2.

On introduction of the abbreviations 39, equations 38 become

$$\left.\begin{aligned} I_1(s) &= Y_{11}(s)E_1(s) + Y_{12}(s)E_2(s), \\ I_2(s) &= Y_{21}(s)E_1(s) + Y_{22}(s)E_2(s). \end{aligned}\right\} \qquad [40]$$

It is seen that it takes three system functions $Y_{11}(s)$, $Y_{22}(s)$, and $Y_{21}(s)$ and two excitation functions $E_1(s)$ and $E_2(s)$ to determine completely the two response transforms $I_1(s)$ and $I_2(s)$ of this two-loop network.

The indicated $\mathfrak{L}^{-1}$ transformation of equations 40 yields, for $0 \leqq t$,

$$\left.\begin{aligned} i_1(t) &(=) \mathfrak{L}^{-1}[Y_{11}(s)E_1(s) + Y_{12}(s)E_2(s)], \\ i_2(t) &(=) \mathfrak{L}^{-1}[Y_{21}(s)E_1(s) + Y_{22}(s)E_2(s)]. \end{aligned}\right\} \qquad [41]$$

The result will be left as two indicated $\mathfrak{L}^{-1}$ transformations.

8. Equations for Two-Loop Network with Mutual Inductance

In the previous two sections it was shown how in an electric network the initial conditions resulting from initially energized self-inductances and capacitances are brought into the transform equations. In the present section the effect of an initially energized mutual inductance will be considered, using as an example the network of Fig. 5·4. This is the network for which the i-d equations were written in Sec. 7, Chapter 2. The driving voltage is $V_m \cos(\omega_1 t + \psi)$. The magnitude of the initial current in L_2 and L_3 is ρ, with direction as indicated. The $\mathfrak{L}$ transforms of the loop currents will now be found.

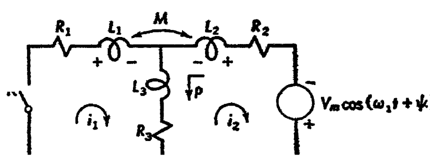


FIG. 5·4

The equations for this network were given in equations 21, Chapter 2, and are repeated here, with the correct signs selected for the mutual-induction terms.

$$(L_1 + L_3)\frac{di_1}{dt} + (R_1 + R_3)i_1 - (L_3 + M)\frac{di_2}{dt} - R_3 i_2 = 0,$$

$$-(L_3 + M)\frac{di_1}{dt} - R_3 i_1 + (L_2 + L_3)\frac{di_2}{dt} + (R_2 + R_3)i_2 \qquad [42]$$

$$= V_m \cos(\omega_1 t + \psi).$$

The initial conditions are $i_1(0) = 0$, and $i_2(0) = -\rho$.

By use of equation 20, Chapter 4, the

$$\mathfrak{L}[V_m \cos(\omega_1 t + \psi)] = \frac{gs + h\omega_1}{s^2 + \omega_1^2} \triangleq V(s),$$

in which $g \triangleq V_m \cos\psi$, and $h \triangleq -V_m \sin\psi$.

Letting the $\mathfrak{L}$ transforms of $i_1(t)$ and $i_2(t)$ be denoted by $I_1(s)$ and $I_2(s)$, the $\mathfrak{L}$ transformation of equations 42 gives

$$\left.\begin{aligned} &(L_1 + L_3)sI_1(s) + (R_1 + R_3)I_1(s) - (L_3 + M)[sI_2(s) - i_2(0)] \\ &\qquad\qquad\qquad\qquad - R_3I_2(s) = 0, \\ &-(L_3 + M)sI_1(s) - R_3I_1(s) + (L_2 + L_3)[sI_2(s) - i_2(0)] \\ &\qquad\qquad\qquad\qquad + (R_2 + R_3)I_2(s) = V(s). \end{aligned}\right\} \quad [43]$$

Terms in $i_1(0)$ have been omitted because they are zero. Collecting terms, and using the fact that $i_2(0) = -\rho$, equations 43 become

$$\left.\begin{aligned} &[(L_1 + L_3)s + R_1 + R_3]I_1(s) - [(L_3 + M)s + R_3]I_2(s) \\ &\qquad\qquad\qquad = (L_3 + M)\rho, \\ &-[(L_3 + M)s + R_3]I_1(s) + [(L_2 + L_3)s + R_2 + R_3]I_2(s) \\ &\qquad\qquad\qquad = V(s) - (L_2 + L_3)\rho. \end{aligned}\right\} \quad [44]$$

For this network, the self- and mutual-impedance functions for the loops are

$$\left.\begin{aligned} z_{11}(s) &\triangleq (L_1 + L_3)s + R_1 + R_3, \\ z_{22}(s) &\triangleq (L_2 + L_3)s + R_2 + R_3, \\ z_{12}(s) = z_{21}(s) &\triangleq -[(L_3 + M)s + R_3]. \end{aligned}\right\} \quad [45]$$

The loop-excitation functions are

$$\left.\begin{aligned} E_1(s) &\triangleq (L_3 + M)\rho, \\ E_2(s) &\triangleq V(s) - (L_2 + L_3)\rho. \end{aligned}\right\} \quad [46]$$

To the rules for forming these functions given in the previous section there can be added now the rule of signs for initial-current-mutual-inductance products. The sign of the $i_2(0)M$ product in $E_1(s)$ will be $+$ if the loop-2 initial current, in subsiding to zero, induces a voltage rise in loop 1 in the arrow direction. A similar rule holds for the sign of the $i_1(0)M$ product in $E_2(s)$. This rule applies equally well where there are three or more inductively coupled loops. In any case, the algebraic sum of the $i(0)L$ and $i(0)M$ products in the $E(s)$ for any loop is the net number of flux linkages in that loop at $t = 0+$.

Finally, the transform equations are

$$\left.\begin{aligned} z_{11}(s)I_1(s) + z_{12}(s)I_2(s) &= E_1(s), \\ z_{21}(s)I_1(s) + z_{22}(s)I_2(s) &= E_2(s), \end{aligned}\right\} \qquad [47]$$

and the corresponding transform diagram is as shown in Fig. 5·5. The algebraic solution of equations 47 for $I_1(s)$ and $I_2(s)$ is the same as that of equations 37 in Sec. 7.

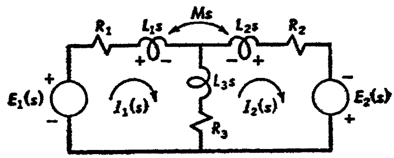


FIG. 5·5. Transform diagram for the network of Fig. 5·4.

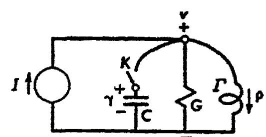


FIG. 5·6

9. EQUATION FOR ONE-NODE-PAIR NETWORK

In the three preceding examples the electric network equations have been formulated on the loop basis. In the present example the $\mathfrak{L}$ transformations will be applied to a network equation formulated on the node basis.

The network of Fig. 5·6 is similar to the one in Fig. 2·13 for which the i-d equation was written in Sec. 11, Chapter 2. Here the closing of the switch K at $t = 0$ connects a capacitance, having initial voltage across it, with the parallel group consisting of an inverse inductance with initial current, a conductance, and a constant driving current I. Let the initial voltage across C have the magnitude γ and polarity shown, and let the initial current in Γ have the magnitude ρ and direction shown. The $\mathfrak{L}$ transform of the node-pair voltage $v(t)$ will be derived.

The equation for this network was given as equation 26, Chapter 2, and is repeated here for convenience,

$$C\frac{dv}{dt} + Gv + \Gamma\int v\,dt = I. \qquad [48]$$

As a result of the initial current ρ in the inductance,

$$\Gamma\int v\,dt = \Gamma\int_0^t vdt + \rho. \qquad [49]$$

The positive sign appears because the initial current in Γ is directed away from the node. Substitution of expression 49 in equation 48 gives

$$C\frac{dv}{dt} + Gv + \Gamma\int_0^t vdt = I - \rho. \qquad [50]$$

Let $V(s)$ be the $\mathfrak{L}$ transform of $v(t)$. Then the $\mathfrak{L}$ transformation of equation 50 gives

$$C[sV(s) - v(0)] + GV(s) + \frac{\Gamma}{s} V(s) = \frac{I}{s} - \frac{\rho}{s}. \qquad [51]$$

But $v(0) = +\gamma$, the sign being $+$ because the initial voltage across C is such as to make the node positive in potential with respect to the reference. Thus equation 51 can be written

$$\left(Cs + G + \frac{\Gamma}{s}\right)V(s) = \frac{I}{s} + \gamma C - \frac{\rho}{s}. \qquad [52]$$

It will be noted that initial condenser charge γC in the node scheme is analogous to initial flux linkages in the loop scheme, and initial current ρ is analogous to initial condenser voltage.

Solving equation 52 for $V(s)$, there is obtained

$$V(s) = \frac{I/s + \gamma C - \rho/s}{Cs + G + \Gamma/s} = \frac{\gamma Cs + (I - \rho)}{Cs^2 + Gs + \Gamma}. \qquad [53]$$

The indicated $\mathfrak{L}^{-1}$ transformation of equation 53 yields, for $0 \leqq t$,

$$v(t) \; (=) \; \mathfrak{L}^{-1}\left[\frac{\gamma Cs + (I - \rho)}{Cs^2 + Gs + \Gamma}\right], \qquad [54]$$

and the result is left in this form.

10. Equations for Two-Coordinate Mechanical System

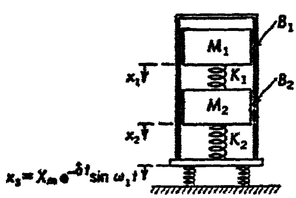


FIG. 5·7. The displacement of the frame is a forced damped oscillation.

The two-coordinate system of Fig. 5·7 will be used to show the application of the $\mathfrak{L}$ transformation to the equations of a mechanical system. This system was described in Sec. 21, Chapter 2. Its differential equations of motion were given in equations 60, Chapter 2. Those equations and the initial conditions supplementing them are repeated below:

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1x_1 - K_1x_2 = B_1 \frac{dx_3}{dt},$$

$$-K_1x_1 + M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + (K_1 + K_2)x_2 \qquad [55]$$

$$B_2 \frac{dx_3}{dt} + K_2x_3,$$

and

$$\left.\begin{aligned} x_1(0) &= a_1, & x_1'(0) &= -b_1, \\ x_2(0) &= -a_2, & x_2'(0) &= b_2. \end{aligned}\right\} \qquad [56]$$

Let the forced displacement $x_3(t) \triangleq X_m e^{-\delta t} \sin \omega_1 t$.

With the aid of pair 6, Table 1, Chapter 4, the

$$\mathfrak{L}[X_m e^{-\delta t} \sin \omega_1 t] = \frac{X_m \omega_1}{(s + \delta)^2 + \omega_1^2} \triangleq X_3(s).$$

If the $\mathfrak{L}$ transforms of x_1 and x_2 are the functions X_1 and X_2 of s, the $\mathfrak{L}$ transformation of equations 55 results in the equations

$$\left.\begin{aligned} &M_1[s^2X_1 - x_1(0)s - x_1'(0)] + B_1[sX_1 - x_1(0)] + K_1X_1 - K_1X_2 \\ &\qquad = B_1[sX_3 - x_3(0)], \\ &-K_1X_1 + M_2[s^2X_2 - x_2(0)s - x_2'(0)] + B_2[sX_2 - x_2(0)] \\ &\qquad + (K_1 + K_2)X_2 = B_2[sX_3 - x_3(0)] + K_2X_3. \end{aligned}\right\} \qquad [57]$$

Collecting terms,

$$\left.\begin{aligned} &(M_1s^2 + B_1s + K_1)X_1 - K_1X_2 = B_1sX_3 + x_1(0)(M_1s + B_1) \\ &\qquad + x_1'(0)M_1 - x_3(0)B_1, \\ &-K_1X_1 + [M_2s^2 + B_2s + (K_1 + K_2)]X_2 = (B_2s + K_2)X_3 \\ &\qquad + x_2(0)(M_2s + B_2) + x_2'(0)M_2 - x_3(0)B_2. \end{aligned}\right\} \qquad [58]$$

Note that the right members of equations 55 were transformed directly instead of first substituting the time expressions for x_3 and its first derivative and then transforming. In this way further advantage is taken of the transformation scheme in that it is much simpler to multiply by the variable in the complex domain than to differentiate in the real domain.

Substitution of the values of the initial conditions can now be made from equations 56. In addition, from the form of $x_3(t)$ it is evident that $x_3(0) = 0$.

With the magnitudes and signs of the initial values supplied, equations 58 can be expressed as

$$\left.\begin{aligned} p_{11}(s)X_1(s) + p_{12}(s)X_2(s) &= E_1(s), \\ p_{21}(s)X_1(s) + p_{22}(s)X_2(s) &= E_2(s), \end{aligned}\right\} \qquad [59]$$

in which

$$
\begin{aligned}
p_{11} &\triangleq M_1s^2 + B_1s + K_1, \\
p_{22} &\triangleq M_2s^2 + B_2s + (K_1 + K_2), \\
p_{12} = p_{21} &\triangleq -K_1, \\
E_1(s) &\triangleq B_1sX_3 + a_1(M_1s + B_1) - b_1M_1, \\
E_2(s) &\triangleq (B_2s + K_2)X_3 - a_2(M_2s + B_2) + b_2M_2.
\end{aligned} \tag{60}
$$

An algebraic solution for the transforms $X_1(s)$ and $X_2(s)$ can now be made, and the inverse transformation of these leads to the final result. Thus the solutions may be indicated as

$$
\begin{aligned}
x_1(t) &(=) \mathfrak{L}^{-1}[X_1(s)], \\
x_2(t) &(=) \mathfrak{L}^{-1}[X_2(s)],
\end{aligned} \tag{61}
$$

and are good for $0 \leqq t$.

11. Set of l Integrodifferential Equations for l-Loop Electric Network

In the preceding examples, no more than two i-d equations or two differential equations have been handled at one time. In this section the more general case of a set of l integrodifferential equations will be treated. These will have first-order derivatives and integrals since equations containing those of higher order arise infrequently and can be reduced to a set of equations of the type treated if encountered.

In an l-loop electric network let the functions $i_1, i_2, \cdots, i_l$ of t be the l unknown loop currents, and let the functions $v_1, v_2, \cdots, v_l$ of t be the l known applied loop voltages. The clockwise direction for a loop current will be taken as positive. The set of i-d equations is

$$\sum_{k=1}^{l} a_{jk}i_k(t) = v_j(t), \quad j = 1, 2, \cdots, l, \tag{62}$$

in which

$$
\begin{aligned}
a_{jj} &\triangleq L_{jj}\frac{d}{dt} + R_{jj} + \frac{1}{C_{jj}}\int dt, \\
a_{jk} &\triangleq -L_{jk}\frac{d}{dt} - R_{jk} - \frac{1}{C_{jk}}\int dt, \quad j \neq k.
\end{aligned}
$$

Certain exceptions to the signs taken for a_{jk} have been discussed in Sec. 9, Chapter 2.

For generality let it be assumed that each loop j has a net initial flux linkage λ_j and a net initial condenser voltage γ_j. In Secs. 7 and 8 a

procedure was developed for handling the initial flux linkages and initial condenser voltages of any loop, and that procedure will be followed here.

Let

$$\mathfrak{L}[i_k(t)] \triangleq I_k(s), \quad \text{and} \quad \mathfrak{L}[v_j(t)] \triangleq V_j(s), \qquad k, j = 1, 2, \cdots, l,$$

then the $\mathfrak{L}$ transformation of equations 62 gives

$$\sum_{k=1}^{l} z_{jk}(s) I_k(s) = E_j(s), \qquad j = 1, 2, \cdots, l, \tag{63}$$

in which

$$z_{jj}(s) \triangleq L_{jj}s + R_{jj} + \frac{1}{C_{jj}s},$$

$$z_{jk}(s) \triangleq -L_{jk}s - R_{jk} - \frac{1}{C_{jk}s}, \quad j \neq k,$$

$$E_j(s) \triangleq V_j(s) + \lambda_j + \frac{\gamma_j}{s},$$

$$\lambda_j \triangleq \text{net flux linkages in loop } j \text{ at } t = 0+,$$

$$\gamma_j \triangleq \text{net condenser voltage in loop } j \text{ at } t = 0+.$$

The set of transform equations 63 includes all the essential information regarding the network whose electrical behavior was described by equations 62. It includes all the essential element constants and element connections, and all the excitations — which in turn include all driving functions and all essential initial values. In other words, this set of equations comprises a complete statement in $\mathfrak{L}$-transform notation of the physical problem that originally was expressed in the set of i-d equations supplemented by a statement of initial conditions.

12. Algebraic Solution of a Set of l Transform Equations

Equations 63 constitute a set of l algebraic equations having l unknown functions $I_1, I_2, \cdots, I_l$ of s. The simplest way of indicating the solution of such a set of equations for any one of the unknowns, say $I_k(s)$, is to use determinants and Cramer's rule. To this end the following notation is introduced:

$$\Delta(s) \triangleq \begin{vmatrix} z_{11}(s) & z_{12}(s) \cdots z_{1l}(s) \\ z_{21}(s) & z_{22}(s) \cdots z_{2l}(s) \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ z_{l1}(s) & z_{l2}(s) \cdots z_{ll}(s) \end{vmatrix}. \tag{64}$$

$\Delta(s)$ is the *determinant of the set* of transform equations and is of order l. The z's are its *elements*. The determinant which is formed when the elements of the kth column of 64 are replaced by the excitation functions $E_1, E_2, \cdots, E_l$ of s is designated by $\Delta_k(s)$. Then by Cramer's rule the solution of the set of equations 63 is

$$I_k(s) = \frac{\Delta_k(s)}{\Delta(s)}, \qquad k = 1, 2, \cdots, l. \tag{65}$$

A determinant can be expanded in terms of the cofactors of the elements of any column or row. This provides a way of carrying the solution of equations 65 somewhat farther. The *cofactor* of the element z_{jk} in $\Delta(s)$ is

$$N_{jk}(s) \triangleq (-1)^{j+k} \begin{vmatrix} z_{11} & \cdots z_{1,k-1} & z_{1,k+1} & \cdots z_{1l} \\ \\ z_{j-1,1} & \cdots z_{j-1,k-1} & z_{j-1,k+1} & \cdots z_{j-1,l} \\ z_{j+1,1} & \cdots z_{j+1,k-1} & z_{j+1,k+1} & \cdots z_{j+1,l} \\ \\ z_{l1} & \cdots z_{l,k-1} & z_{l,k+1} & \cdots z_{ll} \end{vmatrix} \tag{66}$$

in which the argument s of the z's is omitted for simplification. $N_{jk}(s)$ is formed from $\Delta(s)$ by striking out the row and column containing the element z_{jk} and prefixing the sign factor $(-1)^{j+k}$.

Expanding $\Delta_k(s)$ in terms of the cofactors of the kth column — remembering that the elements of this column consist of the E's — equations 65 can be written

$$I_k(s) = \frac{N_{1k}(s)}{\Delta(s)} E_1(s) + \frac{N_{2k}(s)}{\Delta(s)} E_2(s) + \cdots + \frac{N_{lk}(s)}{\Delta(s)} E_l(s),$$
$$k = 1, 2, \cdots, l. \tag{67}$$

When a cofactor is divided by the determinant of the system as above, it is called a *normalized cofactor*. Introducing the abbreviations

$$Y_{kj}(s) \triangleq \frac{N_{jk}(s)}{\Delta(s)} \tag{68}$$

for the normalized cofactors forming the coefficients in the equations 67, these equations become

$$I_k(s) = \sum_{j=1}^{l} Y_{kj}(s) E_j(s), \quad k = 1, 2, \cdots, l. \tag{69}$$

$Y_{jj}(s)$ is the *short-circuit input-admittance function* of the network viewed from loop j. The single transform equation

$$I_j(s) = Y_{jj}(s) E_j(s) \tag{70}$$

indicates that $Y_{jj}(s)$ is the system function relating the transform for the current in loop j to the excitation function for this same loop, all other excitation functions being zero. $Y_{jj}(s)$ can be formed from the determinant and cofactor as in equation 68, but equation 70 indicates that it can also be formed readily from inspection of the transform diagram of the network. In the latter method, impedance functions are combined in series and parallel, with ΔY, or $Y\Delta$ transformations made when necessary as with steady-state impedances.

$Y_{kj}(s)$ is the *short-circuit transfer-admittance function* for loops j and k. The equation

$$I_k(s) = Y_{kj}(s)E_j(s) \qquad [71]$$

indicates that $Y_{kj}(s)$ is the system function relating the transform for the current in loop k to the excitation function in loop j, all other excitation functions being zero. As suggested by equation 71, this function can be formed directly from inspection of the transform diagram of the network by series-parallel combination of impedance functions.

Through choice of symbols and terminology an effort has been made to emphasize the fact that $\mathfrak{L}$-transform relations are but a generalization of conventional steady-state a-c relations. Network theory is well known in terms of steady-state a-c quantities but not in terms of $\mathfrak{L}$ transforms. The similarity of form of the equations is evident, but the nature of the transforms makes possible a generality of interpretation in transient as well as steady-state analysis that far exceeds conventional treatment of these two phases of network analysis. On the other hand, a knowledge of steady-state a-c theory can be of real assistance in the rapid formulation of system functions in the $\mathfrak{L}$-transformation theory. A similarity of symbolism and terminology has been employed to encourage this close association and mutual helpfulness of the two approaches.

PROBLEMS

5·1. When not in use the relay K_2 shown in the diagram is closed. The winding controlling K_2 is divided, winding 1 having few turns N_1 and low resistance, and winding 2 having many turns N_2 and high resistance. When switch K_1 is closed large current is provided for quick pickup of K_2 and low current is provided ultimately for holding it open. When switch K_1 is opened there is a time delay before K_2 closes. The pickup and drop-out ampere-turns are A_1 and A_2, respectively, with $A_2 < A_1$. The supply voltage is V volts dc.

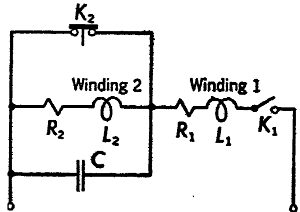


FIG. 5·P1

(a) If switch K_1 is closed and t is considered zero when K_2 opens, form the $\mathfrak{L}$ transform of the current in winding 2.

(b) With the network in its steady-state condition, switch K_1 is opened. If this is considered to occur at $t = 0$, form the $\mathfrak{L}$ transform of the current in winding 2.

5·2. With K in position a, the network illustrated comes to its steady-state condition; K is then moved to position b at $t = 0$. Give the $\mathfrak{L}$ transform of the voltage v_2 for $0 \leq t$.

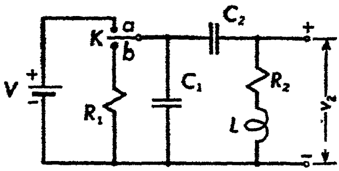


FIG. 5·P2

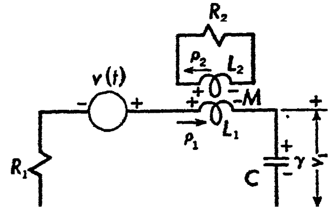


FIG. 5·P3

5·3. Find the $\mathfrak{L}$ transform of the condenser voltage v_1 in the network illustrated. The initial conditions are as indicated. $v(t) \triangleq V_m \sin \omega_1 t$.

5·4. If the voltage source and series branch in diagram a are to be replaced by a current source and parallel branch as in b, what must the parallel branch be, and what relation must hold between the current and voltage sources? Terminal current i_1 and voltage v_1 must be the same in both networks.

Try generalizing this to branches consisting of general passive networks.

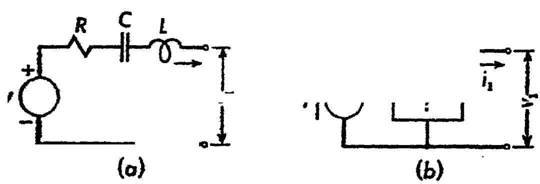


FIG. 5·P4

5·5. If a unit step voltage is impressed on a 2-terminal network which is initially without currents or charges, the input current is

$$i(t) = K_1 + K_2 e^{-\alpha t} \sin \beta t \text{ amperes.}$$

Here K_1 and K_2 are real constants. From this calculate what the steady-state input power will be if the applied voltage on these two terminals is

$$v(t) = V_1 \sin \omega_1 t + 0.2 V_1 \sin (3\omega_1 t + \psi) \text{ volts,}$$

i.e., a fundamental and a 20-percent third harmonic.

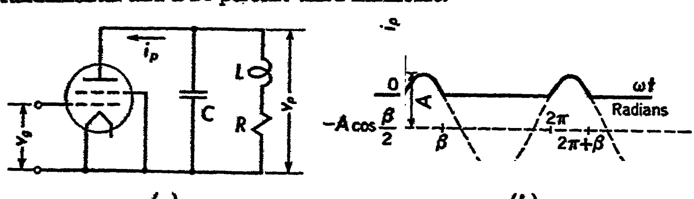


FIG. 5·P6

5·6. In diagram a is shown a screen-grid tube driving a tuned circuit. As a result of a grid-bias voltage, the alternating voltage v_g in the grid circuit produces an inter-

mittent plate current i_p of the form shown by the solid line in diagram b, the tube acting as a current source. Give the $\mathfrak{L}$ transform of the voltage drop v_p produced across the plate circuit by this current.

5·7. A surge-voltage generator which is especially useful for studying the surge protection of transmission-line terminal apparatus is shown in diagram a. When this surge generator discharges without any load between terminals mn, the voltage drop $v(t)$ appearing between these terminals has the form $V_m(e^{-at} - e^{-bt})$, in which V_m, a, and b are positive real numbers. Furthermore, when viewed from these terminals during discharge the surge generator appears to have an impedance function which is a constant R. Thus, viewed from terminals mn, the generator's equivalent network is as shown in diagram b.

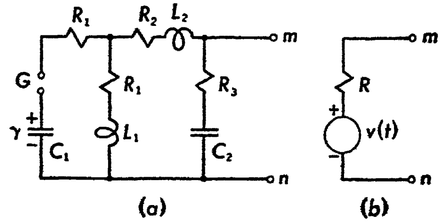


FIG. 5·P7

If C_1, C_2, L_1, and L_2 are fixed in value, find the relations that must hold among the R's and the C's and L's of diagram a to give the $v(t)$ and R of diagram b. Assume that the initial energies are zero except in C_1, the initial voltage across which is γ.

5·8. The network shown in the diagram is in the steady state when switch K opens. Find the $\mathfrak{L}$ transform of the voltage which appears thereafter across the mid-condenser C. The result may be left in the form of a ratio of determinants.

$$V_0 = 300 \text{ volts} \qquad L = 2 \times 10^{-3} \text{ henry}$$
$$R = 10 \text{ ohms} \qquad C = 5 \times 10^{-9} \text{ farad}$$

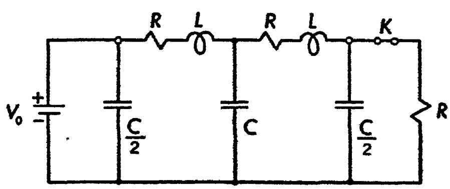


FIG. 5·P8

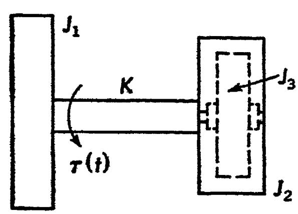


FIG. 5·P9

5·9. The diagram shows a shaft having at one end a disk with moment of inertia J_1 and at the other end a vibration damper consisting of a copper drum whose moment of inertia is J_2 and within which is a flywheel having moment of inertia J_3. This inner flywheel is made of an aluminum-nickel-cobalt-iron alloy having high magnetic retentivity so that it remains polarized after magnetization. The damping produced by eddy currents induced in the drum when there is relative motion between drum and flywheel is represented by rotational resistance B. The torsional stiffness of the shaft is K.

With the system at rest an exciting torque $\tau(t) = T_m \cos \omega_1 t$ is applied to the disk at $t = 0$. Find the $\mathfrak{L}$ transform of the angular twist to which the shaft between the disk and the damper is subjected. In a single system of units,

$J_1 = 100$	$K = 50$	$\omega_1 = 3$
$J_2 = 8$	$B = 1.2$	
$J_3 = 40$	$T_m = 1.2$	

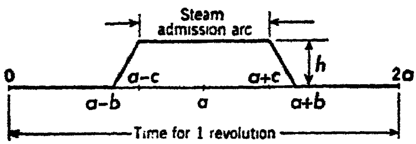


FIG. 5·P10

5·10. The fundamental equation of motion of a turbine blade subject to pulse loading because of partial admission of steam is

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t),$$

in which $f(t)$ is the steam jet loading. The form of one period of $f(t)$ is shown in the diagram.

(a) Find the $\mathfrak{L}$ transform of x for an interval corresponding to one revolution assuming that the initial values of x and x' are known.

(b) If the initial values were not known how would you find the $\mathfrak{L}$ transform for x considering that the blade motion has reached the repeated-transient state?

5·11. The diagram shows the network of a parallel inverter with resistance load. Consider one half-cycle of its operation, letting $t = 0$ at the instant when anode 2 ceases conducting and the return current passes through anode 1 alone. Assume that at that instant the initial condenser voltage and the initial inductance currents are as indicated. The constant arc-voltage drop has been deducted from the source

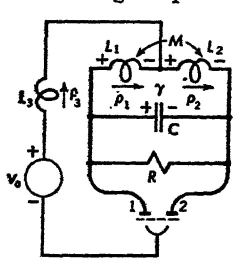


FIG. 5·P11

voltage to give the constant voltage V_0. To simplify the figure, the grid-control circuits are not shown and need not be considered in the problem. For this half-cycle find:

(a) The self-impedance and mutual-impedance functions for the loops.

(b) The short-circuit input-admittance and transfer-admittance functions for the loops.

(c) The excitation functions for the loops.

(d) The $\mathfrak{L}$ transforms of the loop currents.

5·12. The diagram shows one section of a balanced lattice-type corrective network with resistance terminations. Assume that the initial energy storage is zero. $R^2 = L/C$.

(*a*) Find: (1) the $\mathcal{L}$ transform of the input current i_1, (2) the $\mathcal{L}$ transform of the output current i_2, (3) the short-circuit input-admittance function, and (4) the short-circuit transfer-admittance function.

(*b*) Repeat part *a* using two sections. It will be sufficient if this is answered by reasoning physically, using the results obtained from a single section.

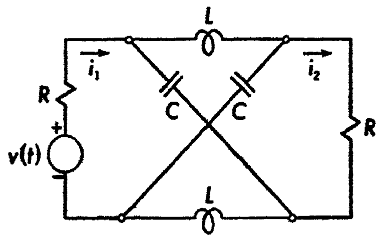


FIG. 5·P12

(*c*) Write the ratio of the transforms of the output and input voltages when there are *n* sections. Examine this ratio along the imaginary axis in the *s*-plane, plotting the magnitude and phase angle of the ratio against ω. In view of the result found here, state in words the interesting electrical properties of this network.

Chapter VI

THE $\mathfrak{L}^{-1}$ TRANSFORMATION OF RATIONAL ALGEBRAIC FRACTIONS

It has been shown in Chapter 5 that the $\mathfrak{L}$ transformation of a set of n linear constant-coefficient i-d equations in one independent variable leads to a set of n linear algebraic equations. The solution of this set for the transform of any one of the unknowns has in general the form

$$\begin{pmatrix}\text{a response}\\ \text{transform}\end{pmatrix}_k = \sum_{j=1}^{n} \begin{pmatrix}\text{a system}\\ \text{function}\end{pmatrix}_{kj} \cdot \begin{pmatrix}\text{an excitation}\\ \text{function}\end{pmatrix}_j, \quad k = 1, 2, \quad , n.$$

The system functions appearing in the terms of the above sum are algebraic functions of s. The same is true of the excitation functions, provided the driving functions are constants, exponentials, sinusoids, positive integral powers of the independent variable, or any product or sum of these — in brief, any functions of the form of pairs 1 to 12 given in Table 1, Chapter 4. Since the product of two rational algebraic functions is also a rational algebraic function, the response transform is a function of the rational algebraic type.

The final step in the solution of a set of linear, constant-coefficient i-d equations with given boundary conditions depends upon the inverse transformation of the various response transforms. With the restriction on the driving function cited above, the problem of obtaining a solution reduces ultimately to the $\mathfrak{L}^{-1}$ transformation of rational algebraic functions of s.

1. Associating Form of $f(t)$ with Position of Poles of $F(s)$

Before discussing the general algebraic fraction it is desirable to review the simple transformation pairs given in Table 1, Chapter 4, since these provide information that can be especially useful in the interpretation of transforms arising in the solution of linear constant-coefficient i-d equations in one independent variable. In the comments that follow, the position given for a pole will refer to its position in the complex plane; the form given for an inverse transform will refer to its form in the range $0 \leq t$.

Pair 1 shows that a first-order pole at the origin is associated with a constant $(0 \leq t)$ in the real domain.

From pair 2 it is seen that a first-order pole on the negative real axis has associated with it a decreasing exponential in the real domain. As the position of this pole is moved in along the negative real axis to the origin and out along the positive real axis the form of the associated time function changes from a decreasing exponential to a constant and then to an increasing exponential.

Pairs 3, 4, and 5 show that first-order conjugate poles on the axis of imaginaries are associated with sinusoidal functions. If these conjugate poles lie off the axis in the left half of the complex plane, as in pairs 6, 7, and 8, the associated function is an exponentially damped oscillation. If these poles lie in the right half-plane the associated function is an exponentially increasing oscillation.

From pair 9 it is seen that a second-order pole at the origin has for its associated function in the real domain a linear; and if this pole is of higher order, it is seen from pair 10 that the associated function is a higher-degree power function.

If the pole is of second order and lies on the negative real axis, pair 11 shows that the associated function in the real domain is the product of a linear factor and a decreasing exponential. If this pole is of higher order, pair 12 shows that the product curve is composed of a power function and a decreasing exponential. Both pairs 11 and 12 show that if the higher-order pole falls on the positive real axis the exponential factor in the associated product function is increasing in magnitude.

It is instructive to follow in this way the change in form of the associated time function as the positions of the poles of $F(s)$ are changed in some prescribed way. A knowledge of the time functions corresponding to transforms having poles in certain locations in the s-plane will be of value in the use of the $\mathfrak{L}$-transformation method. This affords a way, different from the usual one, of thinking about the characteristics and behavior of physical systems and should strengthen one's insight into the performance to be expected of these systems when subjected to specified excitations. This scheme was introduced by Routh [Ro 2] in his Adams Prize Essay in 1877.

2. General Rational Algebraic Fraction

Let the general rational algebraic fraction be designated by

$$F(s) \triangleq \frac{A(s)}{B(s)} \triangleq \frac{a_p s^p + a_{p-1}s^{p-1} + \cdots + a_1 s + a_0}{s^q + b_{q-1}s^{q-1} + \cdots + b_1 s + b_0}, \qquad [1]$$

in which the a's and b's are real constants, and p and q are positive integers. $A(s)$ and $B(s)$ are polynomials in s, i.e., they are entire (or integral) functions. For convenience the coefficient of the highest

power of s in $B(s)$ has been made unity by division of all coefficients of numerator and denominator by a constant. Two cases may be distinguished, depending upon the relative values of p and q.

a. If $p \geqq q$, then $F(s)$ is an improper fraction. $A(s)$ should first be divided by $B(s)$, the quotient being carried out until a remainder is obtained which is a proper fraction. For example, if $p = q + 1$, division gives

$$\frac{A(s)}{B(s)} = K_1 s + K_0 + \frac{A_1(s)}{B(s)}, \qquad [2]$$

with K_0 and K_1 constants, and $\dfrac{A_1(s)}{B(s)}$ a proper fraction. Then

$$\mathfrak{L}^{-1}\left[\frac{A(s)}{B(s)}\right] = \mathfrak{L}^{-1}[K_1 s] + \mathfrak{L}^{-1}[K_0] + \mathfrak{L}^{-1}\left[\frac{A_1(s)}{B(s)}\right]. \qquad [3]$$

Two exceptional cases arise here: (1) the inverse transformation of a constant and (2) the inverse transformation of a positive integral power of s. As none of the pairs developed in Chapter 4 sheds any light on the way to handle these cases, the treatment of an $F(s)$ which is an improper fraction will be postponed until Sec. 9, Chapter 8.

b. If $p < q$, then $F(s)$ is a proper fraction. Two sub-cases may be distinguished: (1) the poles of $F(s)$ are all of the first order and (2) some, or all, of the poles are of higher order than the first.

3. $p < q$ AND FIRST-ORDER POLES ONLY

The poles of $F(s)$, equation 1, are located by determining the roots of the equation $B(s) = 0$. Let these q roots be $s_1, s_2, \cdots, s_q$ — no two of which are equal; and assume that $A(s_k) \neq 0$, $k = 1, 2, \cdots, q$. Exclusion of repeated roots assures that $F(s)$ will have first-order poles only; barring s_k from being a zero of $A(s)$ assures that the number of poles that $F(s)$ has is q and not fewer. $F(s)$ can then be written

$$\frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)(s - s_2)\cdots(s - s_q)}. \qquad [4]$$

The rational fraction $A(s)/B(s)$ can be written as a sum of partial fractions,[1] each partial fraction having for its denominator one of the factors of $B(s)$. There will be q of these partial fractions. Letting the K's be the coefficients, as yet undetermined, this expansion gives

$$\frac{A(s)}{B(s)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \cdots + \frac{K_k}{s - s_k} + \cdots + \frac{K_q}{s - s_q}. \qquad [5]$$

[1] See any college algebra text, for example, Fine.

To evaluate the typical coefficient K_k multiply both members of equation 5 by $(s - s_k)$, obtaining

$$\frac{(s - s_k)A(s)}{B(s)} = K_1 \frac{s - s_k}{s - s_1} + K_2 \frac{s - s_k}{s - s_2} + \cdots + K_k + \cdots + K_q \frac{s - s_k}{s - s_q}. \quad [6]$$

In the fraction forming the left member of equation 6 $(s - s_k)$ is a factor of both numerator and denominator and should be divided out. Then letting $s = s_k$, this left member becomes a number, and in the right member all terms except K_k become zero, i.e.,

$$K_k = \left[\frac{(s - s_k)A(s)}{B(s)}\right]_{s=s_k} = \frac{A(s_k)}{(s_k - s_1)(s_k - s_2)\cdots(s_k - s_{k-1})(s_k - s_{k+1})\cdots(s_k - s_q)}. \quad [7]$$

But

$$(s_k - s_1)(s_k - s_2)\cdots(s_k - s_{k-1})(s_k - s_{k+1})\cdots(s_k - s_q) = \left[\frac{d}{ds}B(s)\right]_{s=s_k} \triangleq B'(s_k), \quad [8]$$

so equation 5 can be written

$$\frac{A(s)}{B(s)} = \sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} \frac{1}{s - s_k}. \quad [9]$$

The $\mathfrak{L}^{-1}$ transformation of the fraction $A(s)/B(s)$ can now be carried out, using the partial-fraction expansion given in equation 9.

$$\mathfrak{L}^{-1}\left[\frac{A(s)}{B(s)}\right] = \mathfrak{L}^{-1}\left[\sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} \frac{1}{s - s_k}\right]. \quad [10]$$

By the linearity theorem (Theorem 5) the order of transformation and summation can be changed; consequently

$$\mathfrak{L}^{-1}\left[\sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} \frac{1}{s - s_k}\right] = \sum_{k=1}^{q} \mathfrak{L}^{-1}\left[\frac{A(s_k)}{B'(s_k)} \frac{1}{s - s_k}\right]. \quad [11]$$

The actual problem of inverse transformation is now a simple one; by pair 2, Table 1, Chapter 4,

$$\mathfrak{L}^{-1}\left[\frac{1}{s - s_k}\right] (=) \, e^{s_k t}, \qquad 0 \leq t.$$

Substitution of this result in equation 11 gives the useful formula

$$\mathfrak{L}^{-1}\left[\frac{A(s)}{B(s)}\right] (=) \sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} e^{s_k t}, \qquad 0 \leqq t, \qquad [12]$$

in which $A(s)/B(s)$ is a rational fraction having first-order poles only, and $s_k(k = 1, 2, \cdots, q)$ is a root of the equation $B(s) = 0$.

Example 1. Find $\mathfrak{L}^{-1}\left[\dfrac{a_1 s + a_0}{(s + \alpha_1)(s + \alpha_2)(s + \alpha_3)}\right]$, in which α_1, α_2, and α_3 are real numbers, all different.

By equation 12,

$$\mathfrak{L}^{-1}\left[\frac{a_1 s + a_0}{(s + \alpha_1)(s + \alpha_2)(s + \alpha_3)}\right] (=) K_1 e^{-\alpha_1 t} + K_2 e^{-\alpha_2 t} + K_3 e^{-\alpha_3 t}, \quad 0 \leqq t, \quad [13]$$

in which

$$K_1 \triangleq \left[\frac{a_1 s + a_0}{(s + \alpha_2)(s + \alpha_3)}\right]_{s=-\alpha_1} = \frac{-a_1\alpha_1 + a_0}{(-\alpha_1 + \alpha_2)(-\alpha_1 + \alpha_3)},$$

$$K_2 \triangleq \left[\frac{a_1 s + a_0}{(s + \alpha_1)(s + \alpha_3)}\right]_{s=-\alpha_2} = \frac{-a_1\alpha_2 + a_0}{(-\alpha_2 + \alpha_1)(-\alpha_2 + \alpha_3)},$$

$$K_3 \triangleq \left[\frac{a_1 s + a_0}{(s + \alpha_1)(s + \alpha_2)}\right]_{s=-\alpha_3} = \frac{-a_1\alpha_3 + a_0}{(-\alpha_3 + \alpha_1)(-\alpha_3 + \alpha_2)}.$$

If all the α's are positive, the result 13 is composed of three decreasing exponential functions.

4. Special Case: One of the Poles Lies at Origin

In $A(s)/B(s)$ of equation 4 let $s_1 \triangleq 0$; then

$$\frac{A(s)}{B(s)} = \frac{A(s)}{s(s - s_2)(s - s_3) \cdots (s - s_q)} = \frac{A(s)}{sB_1(s)}, \qquad [14]$$

in which

$$B_1(s) \triangleq \frac{B(s)}{s} = (s - s_2)(s - s_3) \cdots (s - s_q).$$

The form of $A(s)/B(s)$ given in equation 14 occurs frequently. It arises, for example, when the excitation function is a constant and the system function does not have a pole or a zero at $s = 0$.

Without repeating the various steps of the development, the final

result can be written as the following modification of equation 12:

$$\mathcal{L}^{-1}\left[\frac{A(s)}{sB_1(s)}\right](=)\left[\frac{A(s)}{B_1(s)}\right]_{s=0}+\sum_{k=2}^{q}\left[\frac{A(s)}{sB_1'(s)}\right]_{s=s_k}e^{s_kt}$$
$$=\frac{A(0)}{B_1(0)}+\sum_{k=2}^{q}\frac{A(s_k)}{s_kB_1'(s_k)}e^{s_kt},\qquad 0\leqq t. \qquad [15]$$

Example 1. Find $\mathcal{L}^{-1}\left\{\dfrac{a_1s+a_0}{s[(s+\alpha)^2+\beta^2]}\right\}$, in which α and β are real numbers.

Here $B_1(s) = [(s+\alpha)^2+\beta^2]$, $B_1'(s) = 2(s+\alpha)$, and $s_2, s_3 = -\alpha \pm j\beta$. Then by equation 15,

$$\mathcal{L}^{-1}\left\{\frac{a_1s+a_0}{s[(s+\alpha)^2+\beta^2]}\right\}(=)\frac{a_0}{\beta_0^2}+K_2e^{(-\alpha+j\beta)t}+K_3e^{(-\alpha-j\beta)t},\quad 0\leqq t, \qquad [16]$$

in which $\beta_0^2 \triangleq (\beta^2+\alpha^2)$, and

$$K_2 \triangleq \left[\frac{a_1s+a_0}{2s(s+\alpha)}\right]_{s=-\alpha+j\beta}=\frac{a_0-a_1\alpha+ja_1\beta}{2j\beta(-\alpha+j\beta)}$$
$$=\frac{[(a_0-a_1\alpha)^2+a_1^2\beta^2]^{\frac{1}{2}}e^{j\psi_1}}{j2\beta(\alpha^2+\beta^2)^{\frac{1}{2}}e^{j\psi_2}}=\frac{1}{2\beta\beta_0}[(a_0-a_1\alpha)^2+a_1^2\beta^2]^{\frac{1}{2}}e^{j[\psi-(\pi/2)]},$$

$$\psi_1 \triangleq \tan^{-1}\frac{a_1\beta}{a_0-a_1\alpha},\quad \psi_2 \triangleq \tan^{-1}\frac{\beta}{-\alpha},\quad \psi \triangleq \psi_1-\psi_2;$$

$$K_3 \triangleq \left[\frac{a_1s+a_0}{2s(s+\alpha)}\right]_{s=-\alpha-j\beta}=\bar{K}_2.$$

Coefficients K_3 and K_2 are conjugate complex numbers since $-j$ appears in K_3 wherever $+j$ appears in K_2. The final result can be written,

$$\mathcal{L}^{-1}\left\{\frac{a_1s+a_0}{s[(s+\alpha)^2+\beta^2]}\right\}$$
$$(=)\frac{a_0}{\beta_0^2}+\frac{1}{\beta\beta_0}[(a_0-a_1\alpha)^2+a_1^2\beta^2]^{\frac{1}{2}}e^{-\alpha t}\sin(\beta t+\psi),\qquad 0\leqq t. \qquad [17]$$

If α and β are positive, this result is composed of a constant and a damped oscillation.

5. SPECIAL CASE: AT LEAST ONE PAIR OF CONJUGATE POLES LIES ON AXIS OF IMAGINARIES

In $A(s)/B(s)$ of equation 4 let $s_1 \triangleq j\omega_1$ and $s_2 \triangleq -j\omega_1$, then

$$\frac{A(s)}{B(s)}=\frac{A(s)}{(s-j\omega_1)(s+j\omega_1)(s-s_3)(s-s_4)\cdots(s-s_q)}$$
$$=\frac{A(s)}{(s^2+\omega_1^2)B_2(s)}, \qquad [18]$$

in which

$$B_2(s) \triangleq \frac{B(s)}{s^2 + \omega_1^2} = (s - s_3)(s - s_4) \cdots (s - s_q).$$

The form of $A(s)/B(s)$ given in equation 18 arises when the driving function is sinusoidal with angular frequency ω_1, and the system function has no poles or zeros at $\pm j\omega_1$. The transform of a sinusoidal function of angular frequency ω_1 being proportional to $(s^2 + \omega_1^2)^{-1}$, the latter will appear as a factor in the response transform.

The final form can be written as the following modification of equation 12:

$$\begin{aligned}\mathfrak{L}^{-1}\left[\frac{A(s)}{(s^2 + \omega_1^2)B_2(s)}\right] (=) &\left[\frac{A(s)}{(s + j\omega_1)B_2(s)}\right]_{s=j\omega_1} e^{j\omega_1 t} \\ &+ \left[\frac{A(s)}{(s - j\omega_1)B_2(s)}\right]_{s=-j\omega_1} e^{-j\omega_1 t} \\ &+ \sum_{k=3}^{q} \left[\frac{A(s)}{(s^2 + \omega_1^2)B_2'(s)}\right]_{s=s_k} e^{s_k t} \\ &= \frac{A(j\omega_1)}{2j\omega_1 B_2(j\omega_1)} e^{j\omega_1 t} + \frac{A(-j\omega_1)}{-2j\omega_1 B_2(-j\omega_1)} e^{-j\omega_1 t} \\ &+ \sum_{k=3}^{q} \frac{A(s_k)}{(s_k^2 + \omega_1^2)B_2'(s_k)} e^{s_k t}, \qquad 0 \leqq t. \qquad [19]\end{aligned}$$

The first two terms in equation 19 are conjugate complex functions of t and ω_1. Being conjugate, the sum of their imaginary parts is zero and the sum of their real parts is twice the real part of either. Consequently, these two terms combine into a single sinusoidal function. It is preferable for computation, however, to retain this function in complex form, so the joint result of the first two terms in equation 19 will be written

$$\mathfrak{R}\left[\frac{A(j\omega_1)}{j\omega_1 B_2(j\omega_1)} e^{j\omega_1 t}\right],$$

and 19 becomes

$$\begin{aligned}\mathfrak{L}^{-1}\left[\frac{A(s)}{(s^2 + \omega_1^2)B_2(s)}\right] (=) &\,\mathfrak{R}\left[\frac{A(j\omega_1)}{j\omega_1 B_2(j\omega_1)} e^{j\omega_1 t}\right] \\ &+ \sum_{k=3}^{q} \frac{A(s_k)}{(s_k^2 + \omega_1^2)B_2'(s_k)} e^{s_k t}, \qquad 0 \leqq t. \qquad [20]\end{aligned}$$

Note that the $e^{j\omega_1 t}$ term in equation 19 is the one indicated when the real part is to be taken as in equation 20. The reason for the choice of this term, rather than the conjugate term, will be given in Chapter 7, Sec. 2.

Since $\mathcal{R}[2K_1e^{j\omega_1 t}] \equiv \mathcal{I}[2jK_1e^{j\omega_1 t}]$, in which the operator $\mathcal{I}$ means "take the imaginary part of," equation 20 can also be written

$$\mathcal{L}^{-1}\left[\frac{A(s)}{(s^2+\omega_1^2)B_2(s)}\right] (=) \ \mathcal{I}\left[\frac{A(j\omega_1)}{\omega_1 B_2(j\omega_1)}e^{j\omega_1 t}\right] + \sum_{k=3}^{q}\frac{A(s_k)}{(s_k^2+\omega_1^2)B_2'(s_k)}e^{s_k t}, \qquad 0 \leqq t. \qquad [21]$$

In application, this latter form frequently leads to the final result more directly than the form given in 20.

Example 1. Find $\mathcal{L}^{-1}\left[\dfrac{a_1 s + a_0}{(s^2+\omega_1^2)(s+\alpha)}\right]$, in which ω_1 and α are real numbers.

Here $B_2(s) = (s+\alpha)$, $B_2'(s) = 1$, and $s_3 = -\alpha$. Applying equation 21,

$$\mathcal{L}^{-1}\left[\frac{a_1 s + a_0}{(s^2+\omega_1^2)(s+\alpha)}\right] (=) \ \mathcal{I}\left[\frac{a_0 + j\omega_1 a_1}{\omega_1(\alpha + j\omega_1)}e^{j\omega_1 t}\right] + \frac{-a_1\alpha + a_0}{\alpha^2+\omega_1^2}e^{-\alpha t}, \quad 0 \leqq t.$$

Now

$$\frac{a_0 + j\omega_1 a_1}{\omega_1(\alpha + j\omega_1)} = \frac{(a_0^2+\omega_1^2a_1^2)^{\frac{1}{2}}e^{j\psi_1}}{\omega_1(\alpha^2+\omega_1^2)^{\frac{1}{2}}e^{j\psi_2}} = \frac{1}{\omega_1}\left(\frac{a_0^2+\omega_1^2a_1^2}{\alpha^2+\omega_1^2}\right)^{\frac{1}{2}}e^{j\psi},$$

in which

$$\psi_1 \triangleq \tan^{-1}\frac{\omega_1 a_1}{a_0}, \quad \psi_2 \triangleq \tan^{-1}\frac{\omega_1}{\alpha}, \quad \text{and} \quad \psi = \psi_1 - \psi_2.$$

Writing the final result in terms of real functions only,

$$\mathcal{L}^{-1}\left[\frac{a_1 s + a_0}{(s^2+\omega_1^2)(s+\alpha)}\right] (=) \ \frac{1}{\omega_1}\left(\frac{a_0^2+\omega_1^2a_1^2}{\alpha^2+\omega_1^2}\right)^{\frac{1}{2}} \sin(\omega_1 t + \psi) + \frac{a_0 - a_1\alpha}{\alpha^2+\omega_1^2}e^{-\alpha t}, \qquad 0 \leqq t. \qquad [22]$$

If α and ω_1 are positive, this result is a sinusoid oscillating about an axis displaced along an exponentially decaying curve.

6. $p < q$ AND MULTIPLE-ORDER POLES

The previous discussion has been limited to functions $F(s)$, equation 1, having first-order poles only. Consider now functions $F(s)$ which have poles of higher order. The restriction to proper fractions $(p < q)$ is

still retained. Let the equation $B(s) = 0$ have the n distinct roots s_1, $s_2, \cdots, s_n$; moreover, let

$$\begin{aligned} &s_1 \text{ occur } m_1 \text{ times,} \\ &s_2 \text{ occur } m_2 \text{ times,} \\ &\cdots\cdots\cdots\cdots\cdots \\ &s_n \text{ occur } m_n \text{ times,} \end{aligned}$$

with the restriction that $m_1 + m_2 + \cdots + m_n = q$. Also assume that $A(s_k) \neq 0$, $k = 1, 2, \cdots, n$. This exclusion of s_k as a possible zero of $A(s)$ assures that the multiplicity of pole s_k is m_k and not less. The function $F(s)$ can then be written

$$\frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)^{m_1}(s - s_2)^{m_2} \cdots (s - s_n)^{m_n}}. \tag{23}$$

The fraction $A(s)/B(s)$ can be resolved into a sum of partial fractions. For each pole s_k of multiplicity m_k there are m_k partial fractions of the form

$$\frac{K_{k1}}{(s - s_k)^{m_k}}, \quad \frac{K_{k2}}{(s - s_k)^{m_k - 1}}, \cdots, \frac{K_{km_k}}{s - s_k},$$

in which the K's are constants yet to be determined. Thus, the expansion of equation 23 is

$$\begin{aligned} \frac{A(s)}{B(s)} = \frac{K_{11}}{(s - s_1)^{m_1}} &+ \frac{K_{12}}{(s - s_1)^{m_1 - 1}} + \cdots + \frac{K_{1j}}{(s - s_1)^{m_1 - j + 1}} + \cdots \\ &+ \frac{K_{1m_1}}{s - s_1} \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \frac{K_{k1}}{(s - s_k)^{m_k}} + \frac{K_{k2}}{(s - s_k)^{m_k - 1}} + \cdots + \frac{K_{kj}}{(s - s_k)^{m_k - j + 1}} + \cdots \\ &+ \frac{K_{km_k}}{s - s_k} \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ = &\sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{K_{kj}}{(s - s_k)^{m_k - j + 1}}. \end{aligned} \tag{24}$$

In equation 24 the inner sum with the index j accounts for each of the terms associated with a particular pole, i.e., for each of the terms of a particular row in 24; the outer sum with the index k accounts for each of the n poles, i.e., for each of the rows of 24.

To evaluate the K_k coefficients, first multiply both members of equation 24 by $(s - s_k)^{m_k}$, obtaining

$$\frac{(s - s_k)^{m_k}A(s)}{B(s)} = K_{k1} + K_{k2}(s - s_k) + K_{k3}(s - s_k)^2 + \cdots$$
$$+ K_{km_k}(s - s_k)^{m_k-1}$$
$$+ (s - s_k)^{m_k}\left[\frac{K_{11}}{(s - s_1)^{m_1}} + \cdots + \frac{K_{nm_n}}{s - s_n}\right], \qquad [25]$$

in which there has been gathered within the brackets all terms except those with K_k coefficients. In the left member $(s - s_k)^{m_k}$ is also a factor of $B(s)$ and should be divided out. Now letting $s = s_k$, this left member becomes a number, and in the right member all terms except K_{k1} become zero. Although this procedure can be used to evaluate K_{k1}, it alone is insufficient to evaluate any of the remaining K_k coefficients since each of these is multiplied by a positive power of $(s - s_k)$.

If the term K_k in the right member of equation 25 could be eliminated, and also the factor $(s - s_k)$ that multiplies K_{k2}, a repetition of the process of letting $s = s_k$ would evaluate K_{k2}. These eliminations can be accomplished by differentiating equation 25 once with respect to s. Thus,

$$\frac{d}{ds}\frac{(s - s_k)^{m_k}A(s)}{B(s)} = K_{k2} + 2K_{k3}(s - s_k) + \cdots$$
$$+ (m_k - 1)K_{km_k}(s - s_k)^{m_k-2}$$
$$+ \frac{d}{ds}(s - s_k)^{m_k}\left[\frac{K_{11}}{(s - s_1)^{m_1}} + \cdots + \frac{K_{nm_n}}{s - s_n}\right]. \qquad [26]$$

Now letting $s = s_k$, the value of K_{k2} is determined. It is seen that this procedure of differentiating and then placing $s = s_k$ is the same one that is used for determining the coefficients in a power series, and can be continued until the value of each unknown constant K_k is determined. A repetition of this for each of the other distinct factors of $B(s)$ will re-

sult in the evaluation of all the undetermined coefficients of equation 24. A general formula for finding these coefficients can be written as

$$K_{kj} \triangleq \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{ds^{j-1}}\frac{(s-s_k)^{m_k}A(s)}{B(s)}\right]_{s=s_k}. \qquad [27]$$

The $\mathcal{L}^{-1}$ transformation of the fraction $A(s)/B(s)$ can now be written, using the expansion given in equation 24, i.e.,

$$\mathcal{L}^{-1}\left[\frac{A(s)}{B(s)}\right] = \mathcal{L}^{-1}\left[\sum_{k=1}^{n}\sum_{j=1}^{m_k}\frac{K_{kj}}{(s-s_k)^{m_k-j+1}}\right]. \qquad [28]$$

By the linearity theorem (Theorem 5) the order in which the transformation and double summation are performed may be changed without affecting the result; so

$$\mathcal{L}^{-1}\left[\sum_{k=1}^{n}\sum_{j=1}^{m_k}\frac{K_{kj}}{(s-s_k)^{m_k-j+1}}\right] = \sum_{k=1}^{n}\sum_{j=1}^{m_k}\mathcal{L}^{-1}\left[\frac{K_{kj}}{(s-s_k)^{m_k-j+1}}\right] \qquad [29]$$

Although the original function 23 may be highly complicated because of its generality, the problem of finding its inverse transform is reduced now to the simple transformation indicated in equation 29. By pair 12, Table 1, Chapter 4,

$$\mathcal{L}^{-1}\left[\frac{1}{(s-s_k)^{m_k-j+1}}\right](=)\frac{t^{m_k-j}}{(m_k-j)!}e^{s_k t}, \qquad 0 \leqq t.$$

Substitution of this result in equation 29 gives the formula for the inverse transformation of the general rational algebraic fraction. It is

$$\mathcal{L}^{-1}\left[\frac{A(s)}{B(s)}\right](=)\sum_{k=1}^{n}\sum_{j=1}^{m_k}\frac{K_{kj}}{(m_k-j)!}t^{m_k-j}e^{s_k t}, \qquad 0 \leqq t, \qquad [30]$$

in which

$$K_{kj} \triangleq \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{ds^{j-1}}\frac{(s-s_k)^{m_k}A(s)}{B(s)}\right]_{s=s_k}.$$

Example 1. Find $\mathcal{L}^{-1}\left[\dfrac{a_2s^2+a_1s+a_0}{(s+\alpha)^3s^2}\right]$, in which α is a real number.

This transform has a third-order pole at $-\alpha$ and a second-order pole at the origin. There are two distinct roots, so $n = 2$. If $s_1 \triangleq -\alpha$ and $s_2 \triangleq 0$, then $m_1 = 3$ and $m_2 = 2$.

Rather than use equation 30 directly, it is less confusing to actually write

out the partial-fraction expansion of the function and then transform term by term. Thus

$$\mathfrak{L}^{-1}\left[\frac{a_2s^2 + a_1s + a_0}{(s+\alpha)^3s^2}\right] = \mathfrak{L}^{-1}\left[\frac{K_{11}}{(s+\alpha)^3} + \frac{K_{12}}{(s+\alpha)^2} + \frac{K_{13}}{s+\alpha} + \frac{K_{21}}{s^2} + \frac{K_{22}}{s}\right]$$

$$(=)\left(\frac{K_{11}}{2!}t^2 + K_{12}t + K_{13}\right)e^{-\alpha t} + K_{21}t + K_{22},\ 0 \leqq t, \qquad [31]$$

in which

$$K_{11} \triangleq \left[\frac{a_2s^2 + a_1s + a_0}{s^2}\right]_{s=-\alpha} = \frac{a_2\alpha^2 - a_1\alpha + a_0}{}$$

$$K_{12} \triangleq \left[\frac{d}{ds}\frac{a_2s^2 + a_1s + a_0}{s^2}\right]_{s=-\alpha} = \frac{-a_1\alpha + 2a_0}{\alpha^3},$$

$$K_{13} \triangleq \frac{1}{2!}\left[\frac{d^2}{ds^2}\frac{a_2s^2 + a_1s + a_0}{s^2}\right]_{s=-\alpha} \quad -a_1\alpha + 3a_0$$

$$K_{21} \triangleq \left[\frac{a_2s^2 + a_1s + a_0}{(s+\alpha)^3}\right]_{s=0} = \frac{a_0}{\alpha^3},$$

$$K_{22} \triangleq \left[\frac{d}{ds}\frac{a_2s^2 + a_1s + a_0}{(s+\alpha)^3}\right]_{s=0} \quad \frac{a_1\alpha - 3a_0}{\alpha^4}.$$

7. SUMMARY OF PAIRS FOR $\mathfrak{L}^{-1}$ TRANSFORMATION OF THE GENERAL RATIONAL ALGEBRAIC FRACTION

The important formulas which have been developed in this chapter for the $\mathfrak{L}^{-1}$ transformation of the general rational algebraic fraction are brought together as transform pairs in Table 1.

Pair 1 gives the $\mathfrak{L}^{-1}$ transform of a fraction having only first-order poles. It is the one most frequently used in solving linear constant-coefficient i-d equations. Pairs 1-*a* and 1-*b* are special cases and are included because the step function and the sectioned sine wave — the driving-force functions most frequently met — usually produce response transforms of these two sub-types. These pairs also are closest to the forms of the partial-fraction expansion theorem which are current in the literature of the Cauchy-Heaviside operational calculus. Pair 2 is for fractions having multiple-order poles. It gives the $\mathfrak{L}^{-1}$ transform of the most general proper fraction that can arise in solving linear constant-coefficient i-d equations if the driving-force functions are restricted to functions which are themselves free solutions of i-d equations of this type.

TABLE 1. SUMMARY OF PAIRS FOR $\mathfrak{L}^{-1}$ TRANSFORMATION OF GENERAL RATIONAL ALGEBRAIC FRACTIONS

1. First-order poles only, general case:

$$\frac{A(s)}{B(s)} \quad \Bigg| \quad \sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} e^{s_k t}, \qquad 0 \leqq t.$$

(a) First-order poles only, one pole lying at the origin:

$$\frac{A(s)}{sB_1(s)} \quad \Bigg| \quad \frac{A(0)}{B_1(0)} + \sum_{k=2}^{q} \frac{A(s_k)}{s_k B_1'(s_k)} e^{s_k t}, \qquad 0 \leqq t.$$

(b) First-order poles only, one pair lying on axis of imaginaries:

$$\frac{A(s)}{(s^2 + \omega_1^2)B_2(s)} \quad \Bigg| \quad \mathcal{R}\left[\frac{A(j\omega_1)}{j\omega_1 B_2(j\omega_1)} e^{j\omega_1 t}\right] + \sum_{k=3}^{q} \frac{A(s_k)}{(s_k^2 + \omega_1^2)B_2'(s_k)} e^{s_k t}, \qquad 0 \leqq t$$

or

$$\Bigg| \quad \mathcal{I}\left[\frac{A(j\omega_1)}{\omega_1 B_2(j\omega_1)} e^{j\omega_1 t}\right] + \sum_{k=3}^{q} \frac{A(s_k)}{(s_k^2 + \omega_1^2)B_2'(s_k)} e^{s_k t}, \qquad 0 \leqq t.$$

2. Higher-order poles, general case:

$$\frac{A(s)}{B(s)} \qquad \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{K_{kj}}{(m_k - j)!} t^{m_k - j} e^{s_k t}, \qquad 0 \leqq t$$

$$K_{kj} \triangleq \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} \frac{(s - s_k)^{m_k} A(s)}{B(s)}\right]_{s=s_k}$$

$$B(s) \triangleq (s - s_1)^{m_1}(s - s_2)^{m_2} \cdots (s - s_k)^{m_k} \cdots (s - s_n)^{m_n}$$

$$m_1 + m_2 + \cdots + m_n = q.$$

In Table 1 the fraction $A(s)/B(s)$ is defined to be of the form (or reducible to the form) $\dfrac{a_p s^p + a_{p-1}s^{p-1} + \cdots + a_1 s + a_0}{s^q + b_{q-1}s^{q-1} + \cdots + b_1 s + b_0}$, in which the a's and b's are real constants, p and q are positive integers, and $p < q$. The prime on a function indicates a first derivative of this function with respect to s; for example, $B'(s_k) \triangleq \left[\dfrac{d}{ds} B(s)\right]_{s=s_k}$

8. DETERMINATION OF THE POSITION AND ORDER OF THE POLES

As a preliminary to the expansion of $F(s)$ in partial fractions having s as the variable it is essential that the position and multiplicity of each pole of this function be known. In the foregoing discussion it has been assumed that this information regarding the poles was available, but to obtain this information in treating a physical problem it is necessary to solve an algebraic equation for its roots. This may entail considerable mathematical work. It should be emphasized, however, that this step is not peculiar to the transform method of solving differential equations. It is an inherent part of the task of obtaining, by analytic methods, the

complete solution of linear i-d equations with constant coefficients. In this detail there is no difference between the transform method and other analytic methods of solving such equations. In all analytic methods — as contrasted with machine methods — a knowledge of the factors of the characteristic equation is essential.

For purposes of discussion in this chapter the general transform has been written as the quotient of two polynomials. In the solution of physical problems the response transform consists of the product of a system function and an excitation function. Obviously, it is not necessary to reduce this product to the explicit quotient of two polynomials in order to apply the foregoing theory, since this only makes more difficult the location of the poles of the transform. The determination of the poles belonging to the excitation function is generally only a matter of inspection; in contrast to this, the determination of the poles belonging to the system function usually requires considerable calculation if the system is other than the simplest type.

If the physical system possesses negligible damping for its characteristic vibrations, only the even powers of s in the characteristic equation for the system need to be considered. With damping neglected, the poles of the system function lie on the axis of imaginaries, and the difficulty of finding them from the characteristic equation is much less than the degree of the equation implies, since it can be solved first for squared roots.

As the solution of fourth- and higher-degree algebraic equations with numerical coefficients is practically unavoidable if physical systems other than the simplest are to be handled, it is desirable that the reader acquaint himself — if not already acquainted — with the Graeffe or root-squaring method of solving such equations. Since the underlying theory of this method is not complicated, and the rules of procedure have been explicitly stated, one can learn to use it effectively after a relatively brief period of practice. Since it has been described in a number of places in mathematical and engineering literature [Do 16, Ku 4, Sc 1, Wh 7, Be 3], it will not be treated here. A working knowledge of it will be presumed, however, in a number of the examples and practice problems. Mechanical methods of finding the roots are also available [Di 2].

The solution of third-degree equations does not require a method so general as the Graeffe method. In a third-degree equation there must be at least one real root, and this can be obtained easily by trial using synthetic division, Horner's method, or some modification of these [Hu 4, pp. 76–82]. When found, this real root can be factored out, leaving a quadratic equation whose solution presents no difficulty.

9. Special Procedure when Principal Diagonal of System Determinant has Equal Elements and each Diagonal Paralleling It has Equal Elements.

In general the response transform resulting from the algebraic solution of a set of $\mathfrak{L}$-transformed equations consists of the ratio of two determinants, as shown in Sec. 12, Chapter 5. The usual procedure is to rewrite the system determinant as a polynomial in s, and to factor this polynomial to determine the location and order of the poles belonging to the system. A time-saving modification of this procedure is possible if the principal diagonal of the system determinant is composed of equal self-elements and each of the diagonals paralleling it is composed of equal mutual elements. There is also the special case, represented by the example below, in which *all* the mutual elements are equal [PI 8]. When these conditions hold it is possible to factor the determinant in terms of its elements directly without first reducing it to its polynomial equivalent in s. Furthermore, in making a partial-fraction expansion of the transform, it is possible to simplify the work by using as variable the self-element, which is a *function* of s, rather than s itself.

This special procedure will be illustrated by an example. In a three-loop network let each loop contain self-elements L, R, and C and be coupled to each of the other loops by mutual inductance M; a unit step voltage is applied in loop 1. The initial energy storage is zero. Then

$$z_{11} = z_{22} = z_{33} \triangleq Ls + R + \frac{1}{Cs} \triangleq z_a,$$

$$z_{12} = z_{13} = z_{23} \triangleq Ms \triangleq z_b,$$

$$E_1(s) \triangleq \frac{1}{s}, \quad \text{and} \quad E_2(s) = E_3(s) \triangleq 0.$$

Here all the self-elements are equal, and likewise all the mutual elements are equal. The $\mathfrak{L}$ transform for the current in loop 1 is

$$I_1(s) = \frac{\begin{vmatrix} E_1 & z_b & z_b \\ 0 & z_a & z_b \\ 0 & z_b & z_a \end{vmatrix}}{\begin{vmatrix} z_a & z_b & z_b \\ z_b & z_a & z_b \\ z_b & z_b & z_a \end{vmatrix}} = \frac{(z_a^2 - z_b^2)E_1}{(z_a + 2z_b)(z_a - z_b)^2} = \frac{(z_a + z_b)E_1}{(z_a + 2z_b)(z_a - z_b)} \qquad [32]$$

The steps by which the system determinant is factored are as follows:

$$\begin{vmatrix} z_a & z_b & z_b \\ z_b & z_a & z_b \\ z_b & z_b & z_a \end{vmatrix} = z_a \begin{vmatrix} 1 & \dfrac{z_b}{z_a} & \dfrac{z_b}{z_a} \\ z_b & z_a & z_b \\ z_b & z_b & z_a \end{vmatrix} = \frac{1}{z_a} \begin{vmatrix} 1 & z_b & z_b \\ z_b & z_a^2 & z_a z_b \\ z_b & z_a z_b & z_a^2 \end{vmatrix}$$

$$= \frac{1}{z_a} \begin{vmatrix} 1 & 0 & 0 \\ z_b & z_a^2 - z_b^2 & z_a z_b - z_b^2 \\ z_b & z_a z_b - z_b^2 & z_a^2 - z_b^2 \end{vmatrix}$$

$$= \frac{(z_a - z_b)^2}{z_a} \begin{vmatrix} z_a + z_b & z_b \\ z_b & z_a + z_b \end{vmatrix}$$

$$= (z_a - z_b)^2 (z_a + 2z_b). \qquad [33]$$

Returning now to the expression for $I_1(s)$ and making a partial-fraction expansion, treating the function z_a as variable,

$$I_1(s) = \left(\frac{H_1}{z_a + 2z_b} + \frac{H_2}{z_a - z_b}\right) E_1, \qquad [34]$$

in which

$$H_1 \triangleq \left[\frac{z_a + z_b}{z_a - z_b}\right]_{z_a = -2z_b} = \frac{1}{3},$$

$$H_2 \triangleq \left[\frac{z_a + z_b}{z_a + 2z_b}\right]_{z_a = z_b} = \frac{2}{3}.$$

Here

$$z_a + 2z_b = (L + 2M)s + R + \frac{1}{Cs} = (L + 2M)\,\frac{(s + \alpha_1)^2 + \beta_1^2}{s},$$

$$\alpha_1 \triangleq \frac{R}{2(L + 2M)}, \qquad \beta_1^2 \triangleq \frac{1}{C(L + 2M)} - \alpha_1^2.$$

$$z_a - z_b = (L - M)s + R + \frac{1}{Cs} = (L - M)\,\frac{(s + \alpha_2)^2 + \beta_2^2}{s},$$

$$\alpha_2 \triangleq \frac{R}{2(L - M)}, \qquad \beta_2^2 \triangleq \frac{1}{C(L - M)} - \alpha_2^2.$$

Substitution of these and $1/s$ for E_1 in equation 34 yields

$$I_1(s) = \left[\frac{1}{3(L + 2M)} \cdot \frac{1}{(s + \alpha_1)^2 + \beta_1^2} + \frac{2}{3(L - M)} \cdot \frac{1}{(s + \alpha_2)^2 + \beta_2^2}\right], \qquad [35]$$

and its $\mathfrak{L}^{-1}$ transformation gives

$$i_1(t) \;(=)\; \frac{e^{-\alpha_1 t}\sin \beta_1 t}{3(L+2M)\beta_1} + \frac{2e^{-\alpha_2 t}\sin \beta_2 t}{3(L-M)\beta_2}, \qquad 0 \leqq t. \qquad [36]$$

This network is also of interest from a different point of view. It is an example of a class of networks having the special property that the short-circuit input-admittance function is the same viewed from any loop and the transfer-admittance function is the same for any pair of loops. The analysis of networks of this type is simplified by the use of symmetrical components, since voltage drops of a specified sequence are produced only by currents of that sequence [LY 1, WA 3]. From the point of view of symmetrical components, $z_a + 2z_b$ is the impedance to zero-sequence currents, and $z_a - z_b$ is the impedance to both positive- and negative-sequence currents. The first term of equation 36 is the zero-sequence current; the second term is the sum of the equal positive- and negative-sequence currents. At the outset a separate transform equation could have been written for each of these sequences, and each sequence equation could have been solved by the $\mathfrak{L}^{-1}$ transformation for its current. These sequence components could then have been combined in accordance with the principles of symmetrical components to obtain the actual current. The result would have been the same as that given in equation 36. By this use of symmetrical components, as with the method given above, it is possible to avoid the usual process of factoring a polynomial in s for problems of the type discussed.

PROBLEMS

6-1. Find the $\mathfrak{L}^{-1}$ transforms of the functions given below. α, β, γ, λ, a_1, and a_0 are real numbers.

(a) $\dfrac{s + a_0}{s(s+\alpha)(s+\gamma)}$ (f) $\dfrac{s + a_0}{(s^2+\lambda^2)[(s+\alpha)^2+\beta^2]}$

(b) $\dfrac{s^2 + a_1 s + a_0}{(s+\alpha)(s^2+\beta^2)}$ (g) $\dfrac{s^2 + a_1 s + a_0}{(s+\gamma)(s+\alpha)^2}$

(c) $\dfrac{s^2 + a_1 s + a_0}{(s+\alpha)(s^2+\beta^2)(s^2+\lambda^2)}$ (h) $\dfrac{s + a_0}{(s^2+\beta^2)(s+\alpha)^2}$

(d) $\dfrac{s}{(s+\alpha)^2+\beta^2}$ (i) $\dfrac{s^2 + a_1 s + a_0}{s(s^2+\beta^2)^2}$

(e) $\dfrac{s^2 + a_1 s + a_0}{s[(s+\alpha)^2+\beta^2]}$ (j) $\dfrac{s^2 + a_1 s + a_0}{(s+\gamma)^2[(s+\alpha)^2+\beta^2]}$

6-2. Show that the following constitutes an $\mathfrak{L}$-transform pair:

$$\frac{n!}{s(s+1)(s+2)\cdots(s+n)} \quad \Big| \quad (1-\epsilon^{-t})^n$$

in which n is a non-negative integer.

6-3. Four identical circuits are magnetically coupled as illustrated. A unit step voltage is applied in circuit 1. The initial energy stored in all the circuits is zero.

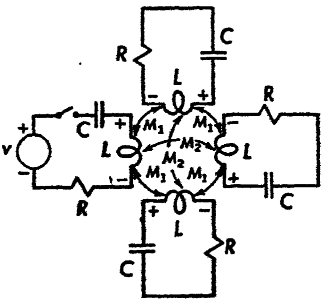


Fig. 6-P3

Assuming that the roots of the characteristic equation are all complex, find the resulting current in circuit 1.

If it is desired to vary the usual procedure in solving this problem, treat it as a problem in transient symmetrical components. Consider the network to be a 4-phase system with the current in each phase composed of symmetrical components. Write the four sequence transform equations by inspection, i.e., one for the zero-sequence components, one for the first-sequence components, etc. Each of these sequence equations can be solved separately by the usual $\mathfrak{L}^{-1}$ transformation. The actual current in any phase can then be found by combining its four sequence components.

Would this use of symmetrical components be advantageous generally in calculating transients in multi-loop networks? Give reasons in support of your answer.

6-4. A lumped-constant representation of a transformer winding is shown in the diagram. One end of the winding is grounded and a unit step voltage is applied

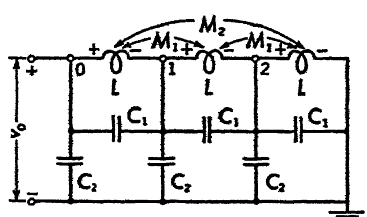


Fig. 6-P4

at the other end. Find the voltage to ground at nodes 1 and 2. The energy stored in the winding at $t = 0$ is zero.

$L = 0.5$ henry $\qquad C_1 = 340 \times 10^{-12}$ farad

$M_1 = 0.2$ henry $\qquad C_2 = 1020 \times 10^{-12}$ farad

$M_2 = 0.1$ henry

CHAPTER VII

THE COMPLETE SOLUTION OF ONE-DIMENSIONAL PROBLEMS CONCERNING ELECTRIC AND MECHANICAL SYSTEMS

Before taking up examples of the complete solution by the $\mathfrak{L}$-transformation method of one-dimensional problems in specific electric and mechanical systems, the solution of a general linear second-order differential equation with constant coefficients will be carried out. The frequent occurrence of equations of this type in many different branches of science and engineering makes its solution of general interest. Furthermore, this solution serves to introduce terminology useful in describing certain important features common to the solutions of many transient problems.

1. SECOND-ORDER DIFFERENTIAL EQUATION

The $\mathfrak{L}$ transformation of a linear second-order differential equation with constant coefficients was carried out in Sec. 4, Chapter 5. The equation used there was

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t), \qquad y \triangleq y(t), \tag{1}$$

and it was found that its indicated solution for $0 \leq t$ is

$$y(t) \; (=) \; \mathfrak{L}^{-1}[Y(s)] = \mathfrak{L}^{-1}\left[\frac{F(s) + y(0)(As + B) + y'(0)A}{As^2 + Bs + C}\right], \tag{2}$$

in which $F(s) \triangleq \mathfrak{L}[f(t)]$, and $y(0)$ and $y'(0)$ are respectively the initial values of y and its first derivative.

In Chapter 6 the basic principles have been presented for carrying out the inverse transformation indicated in equation 2. A driving function $f(t)$ must first be specified; let it be $F_m \cos(\omega_1 t + \psi)$, in which F_m, ω_1, and ψ are real constants. By use of equation 20, Chapter 4,

$$F(s) = \mathfrak{L}[F_m \cos(\omega_1 t + \psi)] = \frac{gs + h\omega_1}{s^2 + \omega_1^2}, \tag{3}$$

in which $g \triangleq F_m \cos\psi$, and $h \triangleq -F_m \sin\psi$. The substitution of expression 3 in $Y(s)$ gives

$$Y(s) = \frac{gs + h\omega_1 + [y(0)(As + B) + y'(0)A](s^2 + \omega_1^2)}{(s^2 + \omega_1^2)(As^2 + Bs + C)}. \tag{4}$$

Note that the numerator has been cleared of fractions. It is essential in the treatment of any algebraic transform first to clear both numerator and denominator of fractions.

The factor $(s^2 + \omega_1^2)$ in the denominator of $Y(s)$ is introduced by the driving function. Let the zeros of this factor be denoted $s_1, s_2 \triangleq \pm j\omega_1$.

The other factor $(As^2 + Bs + C)$ in the denominator is introduced by the left member of the differential equation 1 and is dependent solely upon the system relations described in the equation. It is the characteristic function, defined in Sec. 4, Chapter 5. The equation

$$As^2 + Bs + C = 0,$$

formed by equating this factor to zero, is the *characteristic equation* of the system. Let the roots of this equation be denoted $s_3, s_4 \triangleq -\alpha \pm j\beta$, in which

$$\alpha \triangleq \frac{B}{2A}, \qquad \beta \triangleq \sqrt{\beta_0^2 - \alpha^2}, \quad \text{and} \quad \beta_0^2 \triangleq \frac{C}{A}.$$

The form which the final result takes depends upon whether β^2 is positive, negative, or zero. It will be assumed in this example that $\alpha^2 < \beta_0^2$, making β^2 positive and the corresponding term in the final result a damped oscillation.

The roots of the characteristic equation are called the *characteristic values*. α is called the *damping constant*, β the *characteristic angular frequency*, and β_0 the *characteristic undamped angular frequency*. β_0 is the limiting value of β as the damping approaches zero.

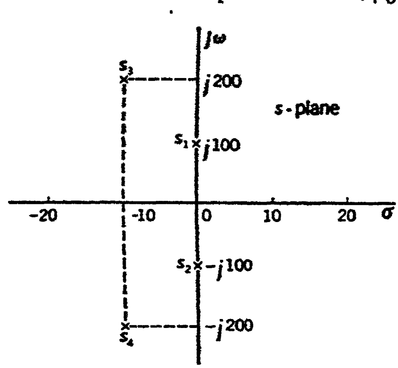


FIG. 7·1. Location of poles of $Y(s)$ assuming $\omega_1 = 100$ radians per second, and $A = 1$, $B = 20$, and $C = 4.01 \times 10^4$ in a single system of units.

The zeros $s_1, s_2, s_3,$ and s_4 of the denominator of $Y(s)$ become the poles of $Y(s)$. In a numerical problem the positions of the poles in the complex plane are determined by the numerical values of the constants ω_1, A, B, C. Here, where the problem is literal, the poles will be assumed to be located in the positions shown in Fig. 7·1.

Inserting the characteristic values in the expression for $Y(s)$, it becomes

$$Y(s) = \frac{gs + h\omega_1 + [y(0)(As + B) + y'(0)A](s^2 + \omega_1^2)}{A(s^2 + \omega_1^2)[(s + \alpha)^2 + \beta^2]}.$$

$Y(s)$ is a proper fraction. It has one pair of conjugate imaginary poles lying on the axis of imaginaries and one pair of conjugate complex poles lying in the left half-plane. It falls within the classification of transforms dealt with in Sec. 5, Chapter 6. Since s_3 and s_4 are conjugate complex numbers and will have associated with them conjugate complex coefficients K_3 and K_4, the solution can be written as

$$\mathcal{L}^{-1}[Y(s)] \;(=)\; \mathcal{R}[2K_1e^{j\omega_1 t}] + \mathcal{R}[2K_3e^{(-\alpha+j\beta)t}], \qquad 0 \leq t, \qquad [5]$$

in which

$$K_1 \triangleq [(s - j\omega_1)Y(s)]_{s_1} = \frac{g - jh}{2A[(\beta_0^2 - \omega_1^2) + j2\alpha\omega_1]}$$

$$= \frac{F_m}{2A[(\beta_0^2 - \omega_1^2)^2 + 4\alpha^2\omega_1^2]^{\frac{1}{2}}}\, e^{j(\psi-\theta)},$$

with

$$F_m e^{j\psi} = g - jh,$$

$$\theta \triangleq \tan^{-1} \frac{2\alpha\omega_1}{\beta_0^2 - \omega_1^2};$$

and

$$K_3 \triangleq [(s + \alpha - j\beta)Y(s)]_{s=-\alpha+j\beta}$$

$$= \frac{1}{2A\beta} \cdot \frac{m - jn}{(\alpha^2 + \omega_1^2 - \beta^2) - j2\alpha\beta}$$

$$= \frac{1}{2A\beta}\left[\frac{m^2 + n^2}{(\beta_0^2 - \omega_1^2)^2 + 4\alpha^2\omega_1^2}\right]^{\frac{1}{2}} e^{j\lambda},$$

with

$$m \triangleq \beta[g + y(0)A(\omega_1^2 - \beta_0^2) - y'(0)B],$$

$$n \triangleq h\omega_1 - g\alpha + y(0)\alpha A(\omega_1^2 + \beta_0^2) + y'(0)A(\alpha^2 + \omega_1^2 - \beta^2),$$

$$\lambda \triangleq \tan^{-1} \frac{-n}{m} - \tan^{-1} \frac{-2\alpha\beta}{\alpha^2 + \omega_1^2 - \beta^2}.$$

The final result is then, in terms of the above abbreviations,

$$y(t) \;(=)\; \frac{F_m}{A[(\beta_0^2 - \omega_1^2)^2 + 4\alpha^2\omega_1^2]^{\frac{1}{2}}} \cos(\omega_1 t + \psi - \theta)$$

$$+ \frac{1}{A\beta}\left[\frac{m^2 + n^2}{(\beta_0^2 - \omega_1^2)^2 + 4\alpha^2\omega_1^2}\right]^{\frac{1}{2}} e^{-\alpha t} \cos(\beta t + \lambda), \qquad 0 \leq t. \qquad [6]$$

The first term in the expression for $y(t)$ has the same sinusoidal waveform and angular frequency ω_1 as the driving function $f(t)$. It is

the *steady-state* portion of the response. It comes from the two terms in the partial-fraction expansion of $Y(s)$ which arise from the conjugate imaginary poles.

The last term in $y(t)$ represents an oscillation with decreasing envelope. With large t this term becomes negligible in comparison with the steady-state term. It is the *transient* portion of the response. It comes from the two terms in the partial-fraction expansion of $Y(s)$ which arise from the conjugate-complex poles in the left half-plane. The magnitude of the real coordinate of each of these two poles is the damping constant α. The reciprocal of the damping constant is the *time constant.* The magnitude of the imaginary coordinate of each of these poles is the characteristic angular frequency β. Both α and β depend solely upon the constants of the physical system and the system's interconnection and are independent of its excitation.

The form of the transient response depends on the position in the complex plane of the poles arising from zeros of the characteristic function. In the present example β^2 has been arbitrarily taken positive so that these two poles arising from the characteristic function fall in the second and third quadrants and yield a transient which is oscillatory with exponentially decreasing envelope. If β^2 were negative, these two poles would fall on the negative real axis at different points, and the transient would consist of the difference of two exponentials, each decreasing at a different rate. It is also possible to have β^2 zero. In this critical case the poles fall on the negative real axis and are coincident, producing one second-order pole. The transient then consists of the product of a decreasing exponential and a linear factor. In all these cases α has been assumed positive, giving positive damping and a transient which diminishes in magnitude and ultimately becomes negligible. The rate at which the transient subsides is dependent directly upon the magnitude of α.

Whereas the character of the transient is determined by the positions of the system-function poles in the complex plane, its amplitude depends upon the initial conditions and the initial phase of the driving function. This can be seen by noticing that m and n are functions of these parameters. It is evident that particularly unfavorable initial conditions and choice of ψ jointly can cause a transient term of appreciable size compared with the size of the steady-state term. Even if the initial conditions were each taken to be zero, there would still be a transient, since there is no value of ψ that makes both m and n zero. For complete absence of a transient, the relations which must hold among the three quantities ψ, $y(0)$, and $y'(0)$ are obtained by setting the expressions for m and n each equal to zero. These derived relations can be anticipated

by reasoning physically. For zero transient the initial conditions $y(0)$ and $y'(0)$ must have the same values as the steady state and its derivative have at the initial phase $(\psi - \theta)$. Since no readjustments in energy balance are then necessary the system can enter immediately into the steady state.

Equation 6 has the form

$$y(t) = y_s(t) + y_t(t)$$

in which $y_s(t)$ is the steady-state component of $y(t)$, and $y_t(t)$ is the transient component. A graphical representation of the instantaneous values of these components and their sum is given in Fig. 7·2 for the particular values of constants shown with this figure and with Fig. 7·1.

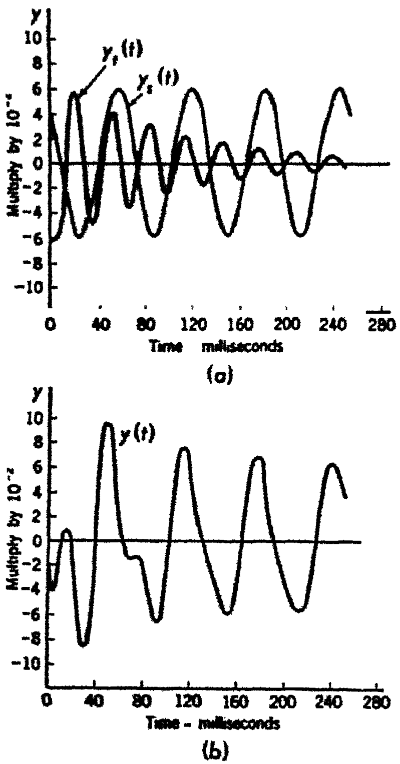


FIG. 7·2. Plot of response $y(t)$, also its steady-state and transient components, assuming

$F_m = 1.81 \times 10^3$ $\quad y(0) = -0.02$
$\Psi = 48.8°$ $\quad y'(0) = -10$

The equation is

$$y(t) = 6.00 \times 10^{-2} \cos(100t + 45°) + 6.97 \times 10^{-2} e^{-10t} \cos(200t + 153°).$$

2. ROTATING PLANE-VECTOR REPRESENTATION OF THE SOLUTION

A vector representation of the components of $y(t)$ can be made using the complex functions which are operated on by $\mathcal{R}$ in equation 5. Thus using the complex function $2K_1e^{j\omega_1 t}$, $y_s(t)$ can be represented as a plane vector of constant magnitude $2\,|K_1|$ rotating counterclockwise with angular velocity ω_1. Its phase at $t = 0$ is $(\psi - \theta)$. This is shown in Fig. 7·3.

Similarly using the complex function $2K_3e^{(-\alpha+j\beta)t}$, $y_t(t)$ can be represented (Fig. 7·3) as a plane vector of decreasing magnitude $2\,|K_3|e^{-\alpha t}$ rotating counterclockwise with angular velocity β. Its phase at $t = 0$ is λ. Whereas the tip of the steady-state vector has for its locus a circle about the origin, the tip of the transient vector has for its locus an exponential spiral about the origin. Vectors of the latter type are sometimes called shrinking rotating vectors.

The vector representation of the functions of this example can be made complete in Fig. 7·3 by representing as a rotating plane-vector the driving function

$$f(t) \triangleq F_m \cos(\omega_1 t + \psi) = \mathcal{R}[F_m e^{j\psi} e^{j\omega_1 t}].$$

The instantaneous projections on the axis of reals of the rotating vectors $2K_1 e^{j\omega_1 t}$ and $2K_3 e^{(-\alpha+j\beta)t}$ give the waves $y_s(t)$ and $y_t(t)$, respectively, shown in Fig. 7·2–*a*. If the vector for $y_t(t)$ is considered to rotate about the tip of the vector for $y_s(t)$, and a resultant vector for the two is drawn from the origin, the instantaneous projection of this resultant on the axis of reals gives the wave $y(t)$ shown in Fig. 7·2–*b*.

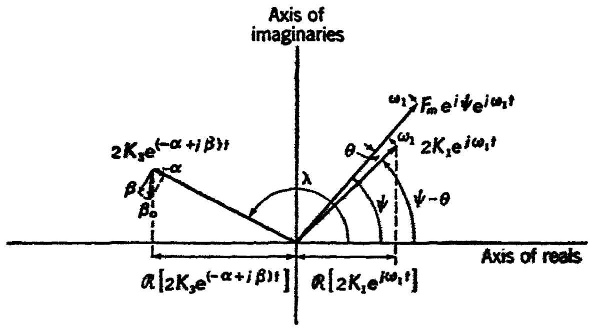


FIG. 7·3. Vector representation of $f(t)$ and the steady-state and transient components of $y(t)$, drawn for instant $t = 0$. The equations are

$$f(t) = \mathcal{R}\,[1.81 \times 10^{3} e^{j48.8^\circ} e^{j100t}],$$
$$y(t) = \mathcal{R}\,[6.00 \times 10^{-2} e^{j45^\circ} e^{j100t}]$$
$$+ \mathcal{R}\,[6.97 \times 10^{-2} e^{j153^\circ} e^{(-10+j200)t}].$$

When the operation of taking the real part of one of two conjugate functions was introduced in Sec. 5, Chapter 6, attention was called to the custom of taking the real part of twice the function having the *positive* imaginary exponent. Now that a vector representation of the complex functions composing a solution has been introduced, the reason for this choice can be seen more clearly. Of any two conjugate exponential functions, the one with positive imaginary exponent yields, on vector representation, the positively rotating vector.

3. Calculation of Steady State

In Sec. 1 the dependence of the steady-state part of the response on the driving function was shown. A more general discussion of the determination of the steady-state part of the response will now be given.

Let $F(s)$ be the transform of the driving function $f(t)$, $G(s)$ be the system function, and $H(s)$ be the transform of the response $h(t)$, then the general transform equation is

$$H(s) = G(s)F(s). \qquad [7]$$

The terms of this equation were defined in Sec. 4, Chapter 5. For purposes here the initial-excitation function is assumed to be zero so that the excitation function becomes simply the driving transform.

If $f(t)$ is a sinusoidal wave, or sum of these, or a constant, the response will have the form

$$h(t) = h_s(t) + h_t(t), \qquad [8]$$

in which $h_s(t)$ and $h_t(t)$ represent the steady-state and transient components, respectively. Either one of these two components can be evaluated without the necessity of considering the other. See also [DR 1, 2]. As a result of this, the $\mathfrak{L}$-transformation method can be used conveniently in strictly steady-state problems.

As an example of the calculation of the steady-state component by this method, let $f(t) \triangleq F_m \sin \omega_1 t$. The transform of $f(t)$ is $F(s) = F_m\omega_1/(s^2 + \omega_1^2)$. Then the steady-state component is

$$h_s(t) = \mathcal{R}[2K_1 e^{j\omega_1 t}] = \mathcal{J}[2jK_1 e^{j\omega_1 t}], \qquad [9]$$

in which

$$K_1 \triangleq [(s - j\omega_1)H(s)]_{s=j\omega_1} = \frac{F_m G(j\omega_1)}{2j}.$$

The complex amplitude of $h_s(t)$ is

$$2K_1 = F_m G(j\omega_1) e^{-j(\pi/2)} = F_m \left| G(j\omega_1) \right| e^{j[\psi-(\pi/2)]} \qquad [10]$$

or

$$2jK_1 = F_m G(j\omega_1) = F_m \left| G(j\omega_1) \right| e^{j\psi}.$$

Carrying out either of the operations indicated in equation 9,

$$h_s(t) = F_m \left| G(j\omega_1) \right| \sin (\omega_1 t + \psi). \qquad [11]$$

This result has been found without considering the transient that may have occurred in reaching this steady state.

4. Three General Problems Expressed in Terms of the General Transform Equation

The general transform equation

$$H(s) = G(s)F(s) \qquad [7]$$

given in Sec. 3 can be used to characterize three general problems. A brief statement of these follows.

a. If $G(s)$ and $F(s)$ are known, equation 7 is an explicit relation for $H(s)$. It expresses the problem of *analysis*, in that the system characteristics and the disturbing force are known and straightforward analysis will give the system's response. Most of the problems considered in this text are of this type.

b. If $F(s)$ and $H(s)$ are known, the system function $G(s)$ is the unknown. That is, equation 7 can be solved for

$$G(s) = \frac{H(s)}{F(s)}. \qquad [12]$$

Equation 12 is the starting point of the *synthesis* problem. This problem is the determination or design of a physical system whose transfer function is $G(s)$. In general the synthesis problem is much more difficult than the problem of analysis and is beyond the scope of this text [LE 1].

c. If $H(s)$ and $G(s)$ are known, the unknown is the excitation function. Equation 7 can be solved for

$$F(s) = \frac{H(s)}{G(s)}. \qquad [13]$$

It is representative of the *system distortion-correction* problem in recording instruments. This problem is the determination of the input disturbance when the output and the distortion characteristics of the system serving as a recording agency are known.

In these preliminaries certain useful terminology has been introduced, the plane-vector representation of the components of the transient solutions has been explained, and the use of transform equations for steady-state calculations and for delineating the three general problems of analysis, synthesis, and distortion correction has been presented. Attention will be given now to the solution of a number of specific examples in different fields of engineering interest.

5. SURGE-VOLTAGE GENERATOR

A surge-voltage generator consists of a number of capacitors which can be charged in parallel and then discharged in series to produce a high-voltage surge. The actual generator network with its charging resistors and discharging spark gaps is one of considerable complexity, but for approximate calculation of the waveform delivered on discharge, these internal complications can be neglected and the generator approximated by a single lumped capacitance in series with inductance and resistance. This series circuit is completed through an external dis-

charge resistor which in turn is paralleled by the apparatus under test. The load on a surge generator usually influences the waveform of the surge to which this load is subjected, so considerable calculation is necessary in the application of such generators [TH 2].

The discharging gap is a nonlinear element in the network, but the varying voltage drop that it introduces is taken as negligible compared with the other voltage drops of the network. The gap is accordingly treated as a switch.

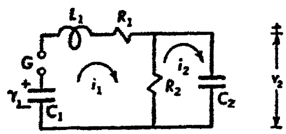


FIG. 7-4. Equivalent network of a surge-voltage generator with transformer load.

$C_1 = 12.5 \times 10^{-3}$ microfarad	$R_2 = 3000$ ohms
$L_1 = 0.25 \times 10^{-3}$ henry	$C_2 = 0.30 \times 10^{-3}$ microfarad
$R_1 = 2000$ ohms	$\gamma_1 = 300$ kilovolts

The equivalent network of a surge-voltage generator serving to test the insulation properties of an open-circuited transformer is shown in Fig. 7-4. For effects during the first few hundred microseconds, the transformer is represented approximately by its equivalent lumped capacitance C_2 to ground [BE 10]. The initial voltage across the generator equivalent capacitance C_1 is γ_1; the initial voltage across C_2 and the initial current in L_1 are zero. A method for calculating the waveform of the surge voltage impressed on the transformer during the first few hundred microseconds after the short-circuiting of the gap G will be given.

Let the functions I_1, I_2, and V_2 of s be the $\mathfrak{L}$ transforms, respectively, of the functions i_1, i_2, and v_2 of t indicated on the diagram. By inspection of the connection diagram, the transform equations can be written directly. They are

$$\begin{aligned} \left(L_1 s + R_1 + R_2 + \frac{1}{C_1 s}\right) I_1(s) - R_2 I_2(s) &= \frac{\gamma_1}{s}, \\ -R_2 I_1(s) + \left(R_2 + \frac{1}{C_2 s}\right) I_2(s) &= 0. \end{aligned} \qquad [14]$$

Solution of these for $I_2(s)$ gives

$$I_2(s) = \frac{\gamma_1 R_2 s}{L_1 R_2 s^3 + \left(\dfrac{L_1}{C_2} + R_1 R_2\right) s^2 + \left(\dfrac{R_1 + R_2}{C_2} + \dfrac{R_2}{C_1}\right) s + \dfrac{1}{C_1 C_2}} \cdot \qquad [15]$$

From this, since the initial voltage across C_2 is zero,

$$V_2(s) = \frac{I_2(s)}{C_2 s} = \frac{a_0}{s^3 + b_2 s^2 + b_1 s + b_0}, \qquad [16]$$

in which, using the numerical values given with Fig. 7·4,

$$a_0 \triangleq \frac{\gamma_1}{L_1 C_2} = 4.0 \times 10^{15}$$

$$b_0 \triangleq \frac{1}{L_1 R_2 C_1 C_2} = 35.6 \times 10^{16}$$

$$b_1 \triangleq \frac{1}{L_1 R_2}\left(\frac{R_1 + R_2}{C_2} + \frac{R_2}{C_1}\right) = 22.6 \times 10^{12}$$

$$b_2 \triangleq \frac{1}{L_1 R_2}\left(\frac{L_1}{C_2} + R_1 R_2\right) = 9.10 \times 10^{6}.$$

To three significant figures the roots of the characteristic equation,

$$s^3 + b_2 s^2 + b_1 s + b_0 = 0, \qquad [17]$$

are found to be

$$s_1 \triangleq -1.59 \times 10^4, \quad \text{and} \quad s_2, s_3 \triangleq (-4.54 \pm j1.29) \times 10^6.$$

The inverse transformation of $V_2(s)$ gives, in accordance with the method presented in Sec. 3, Chapter 6,

$$v_2(t) = \mathfrak{L}^{-1}\left[\frac{a_0}{(s - s_1)(s - s_2)(s - s_3)}\right]$$
$$(=)\ K_1 e^{s_1 t} + \Re[2K_2 e^{s_2 t}], \qquad 0 \leqq t, \qquad [18]$$

in which

$$K_1 \triangleq \left[\frac{a_0}{(s - s_2)(s - s_3)}\right]_{s=s_1} = 181,$$

$$K_2 \triangleq \left[\frac{a_0}{(s - s_1)(s - s_3)}\right]_{s=s_2} = 330e^{j105.9^\circ}.$$

The use of degrees in this way is incorrect but is standard practice in electrical engineering because of the earlier lack of sine, cosine, and tangent tables for angles in radians. Finally, with t in microseconds,

$$v_2(t) = 181e^{-0.0159t}$$
$$+ 660e^{-4.54t} \cos(1.29t + 105.9^\circ) \text{ kv}. \qquad [19]$$

The form of v_2 is shown in Fig. 7·5.

Equation 19 shows that $v_2(t)$ is composed of a decreasing exponential and a damped oscillation. These have the following characteristics:

Time constant of exponential $= 1/(1.59 \times 10^{-2}) = 62.5$ microseconds,
Time constant of damped oscillation $= 1/4.54 = 0.220$ microsecond,
Period of the oscillation $= 2\pi\ 1.29 = 4.87$ microseconds.

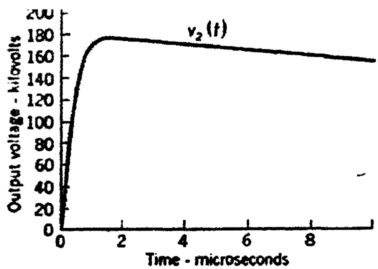


FIG. 7·5. Waveform of surge voltage applied to transformer.

The time constant of the exponential is so large compared with that of the damped oscillation that the sum of the two functions, except for the first half microsecond, is substantially a slowly decaying exponential. The damped oscillation enables $v_2(t)$ to start at zero and yet quickly attain the value required by the exponential term.

6. VACUUM-TUBE AMPLIFIER

Vacuum-tube amplifier networks provide particularly good examples for analysis on the node basis since their equivalent networks include many parallel branches. In general the vacuum tube is a nonlinear device, but by careful design of the network in which it operates as an amplifier it can be made to function as a linear element over a range of frequencies sufficient for most applications. To the extent that it behaves as a linear device, a vacuum-tube amplifier admits of analysis by the methods treated here [JA 1].

Vacuum-tube amplifiers are used widely to amplify transient voltage pulses. Unfortunately, even with the proper selection of voltages and associated network elements to make its vacuum tubes function as linear devices, an amplifier does not amplify transients with absolute fidelity. Distorting surges are introduced by the terminal network elements and by the interelectrode capacitances of the tubes. For best performance these distorting surges should be of short duration compared with the duration of the transient pulses that are being amplified. One of the ways in which the merit of a particular amplifier can

be displayed, either experimentally or analytically, is by its response to a unit step input voltage.

The diagrammatic representation of a triode (a 3-element vacuum tube) is shown in Fig. 7·6–*a*, and a low-frequency equivalent network [Ni 1] is shown in Fig. 7·6–*b*. The variable part of the grid-circuit voltage is v_0; it is

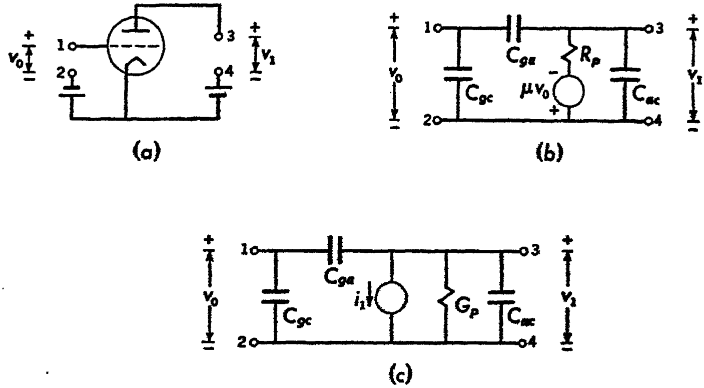


FIG. 7·6. Diagrammatic representation and a low-frequency equivalent network of a triode. In *c*, $i_1 = G_p \mu v_0$ and $G_p = R_p^{-1}$.

the input voltage. The variable part of the plate-circuit voltage is v_1; it is the output or load voltage. In the equivalent network only these variable or incremental voltages are included, since only the variations from the constant grid-circuit voltage and plate-circuit voltage have importance here. C_{gc} is the grid-cathode capacitance, C_{ga} is the grid-anode capacitance, and C_{ac} is the anode-cathode capacitance. R_p is the dynamic internal plate resistance, and μ is the amplification factor.

If the equivalent network is to be treated on the node basis the analysis can be simplified by replacing (see Sec. 14, Chapter 2) the voltage source μv_0 and resistance R_p, which are in series, by a current source i_1 and conductance G_p, which are in parallel, as in Fig. 7·6–*c*.

To show the application of the $\mathfrak{L}$ transformation to the calculation of an amplifier response, two examples will be given. The first treats a simple one-stage amplifier using a triode, and the second treats a multistage amplifier using pentodes (5-element tubes).

a. The equivalent network for a single-stage amplifier using a triode and supplying a resistance load is shown in Fig. 7·7. Let the input voltage $v_0(t)$ be a unit step voltage $u(t)$. The output voltage $v_1(t)$ is to be calculated.

The node-pair voltage v_1 is chosen as the dependent variable, and the

equation is formulated on the node basis. The current equation is

$$C_2 \frac{dv_1}{dt} + C_3 \frac{d}{dt}(v_1 - v_0) + (G_1 + G_p)v_1 = -i_1(t),$$

in which $i_1(t) = G_p\mu v_0(t)$. On rearranging terms, this equation becomes

$$(C_2 + C_3)\frac{dv_1}{dt} + (G_1 + G_p)v_1 = C_3 \frac{dv_0}{dt} - i_1(t). \qquad [20]$$

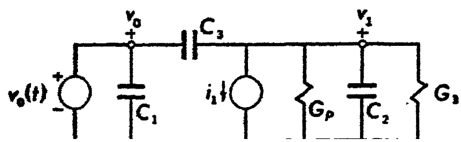


FIG. 7·7. Single-stage triode amplifier with conductance load.

$C_1 = 6 \times 10^{-6}$ microfarad $\quad G_p = 1.7 \times 10^{-5}$ mho
$C_2 = 12 \times 10^{-6}$ microfarad $\quad G_1 = 4 \times 10^{-6}$ mho
$C_3 = 2 \times 10^{-6}$ microfarad $\quad \mu = 100$

Letting $\mathcal{L}[v_1(t)] \triangleq V_1(s)$, and $\mathcal{L}[i_1(t)] \triangleq I_1(s)$, the $\mathcal{L}$ transformation of equation 20 gives

$$[(C_2 + C_3)s + G_1 + G_p]V_1(s)$$
$$= C_3sV_0(s) - I_1(s) + (C_2 + C_3)v_1(0+) - C_3v_0(0+). \qquad [21]$$

As a result of neglecting resistance in the condenser branches there is a network of pure capacitances across the input source, and with a unit step change in input voltage, the voltages across these capacitances will change abruptly at $t = 0$. The initial voltages $v_0(0+)$ and $v_1(0+)$ are consequently 1 and $C_3/(C_2 + C_3)$, respectively. However, for solution of equation 21 it is not necessary actually to evaluate $v_1(0+)$ and $v_2(0+)$. Inspection of the capacitance network shows that if the charge on node 1 is to be conserved the net charge on this node at time $0+$ equals its net charge at $0-$, which means that

$$C_3[v_1(0+) - v_0(0+)] + C_2v_1(0+) = 0,$$

or that

$$(C_2 + C_3)v_1(0+) - C_3v_0(0+) = 0. \qquad [22]$$

Thus in equation 21 the terms containing $v_1(0+)$ and $v_2(0+)$ add to give the initial net charge on node 1, and this is zero.

Solving equation 21 for $V_1(s)$,

$$V_1(s) = \frac{C_3}{C_2 + C_3} \cdot \frac{s - a_0}{s(s + \alpha)}, \qquad [23]$$

in which s^{-1} has replaced $V_0(s)$,

$$a_0 \triangleq \frac{\mu G_p}{C_3}, \quad \text{and} \quad \alpha \triangleq \frac{G_1 + G_p}{C_2 + C_3} = 1.50 \times 10^6.$$

The $\mathfrak{L}^{-1}$ transformation of equation 23 yields for $0 \leqq t$,

$$v_1(t) \; (=) \; K_0 + K_1 e^{-\alpha t}, \tag{24}$$

in which

$$K_0 \triangleq [sV_1(s)]_{s=0} = \frac{C_3}{C_2 + C_3} \cdot \frac{-a_0}{\alpha} = -\mu \frac{G_p}{G_1 + G_p} = -80.9,$$

$$K_1 \triangleq [(s + \alpha)V_1(s)]_{s=-\alpha} = \frac{C_3}{C_2 + C_3} \cdot \frac{\alpha + a_0}{\alpha} = \frac{C_3}{C_2 + C_3} + \mu \frac{G_p}{G_1 + G_p}$$
$$= 81.0.$$

Thus, with t in microseconds,

$$v_1(t) \; (=) \; -80.9 + 81.0e^{-1.50t} \text{ volts}, \qquad 0 \leqq t, \tag{25}$$

and its form is shown in Fig. 7·8. The time constant of $v_1(t)$ is $\alpha^{-1} = 0.67$ microsecond.

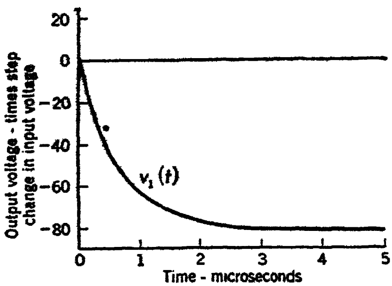


FIG. 7·8. Response of triode amplifier to unit step change of grid voltage.

b. The equivalent network for a pentode vacuum tube is given in Fig. 7·9. In the pentode the additional electrodes act as screens and reduce the grid-anode capacitance to a negligible value. This reduction of the grid-anode capacitance simplifies the calculation of the response of multi-stage amplifiers since it is possible to treat the output of any stage as dependent on the input from the preceding stage but independent of the load on the succeeding stage.

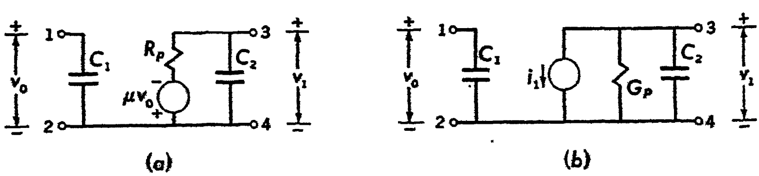


FIG. 7·9. Equivalent network of a pentode. In *b*, $i_1 = G_p \mu v_0$ and $G_p = R_p^{-1}$.

The equivalent network for the first and last stages of an n-stage, RC-coupled amplifier using pentodes is shown in Fig. 7·10. The first

$n - 1$ stages are alike; the nth stage has the conductance load G_l. The grid-circuit conductance is G_2. The coupling between tubes is through capacitance C_3 and conductance G_1.

The output voltage $v_{2n-1}(t)$ resulting from an input $v_0(t) \triangleq u(t)$ will now be calculated. Since there is interest only in the *change* from the steady operating point as a result of this sudden change in the input

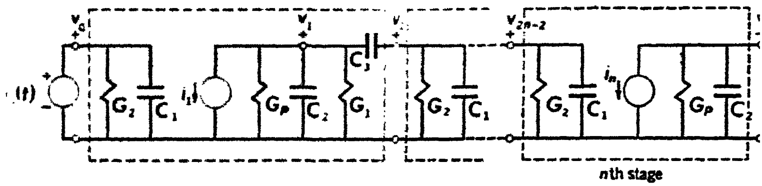


FIG. 7·10. Equivalent network of first and last stages of an GC-coupled pentode amplifier.

$C_1 = 7 \times 10^{-6}$ microfarad	$G_p = 0.7 \times 10^{-4}$ mho	$G_l = 4 \times 10^{-6}$ mho
$C_2 = 12 \times 10^{-6}$ microfarad	$G_1 = 4 \times 10^{-6}$ mho	$\mu = 2000$
$C_3 = 5 \times 10^{-3}$ microfarad	$G_2 = 2 \times 10^{-6}$ mho	$i_k = G_p\mu v_{2k-2}$,
		$k = 1, 2, 3, \cdots, n.$

voltage, the initial condenser voltages can be considered zero. Although C_1 of the first stage is an exception because the source voltage is applied directly across its terminals, this condenser does not enter into the analysis.

If the $\mathfrak{L}$ transforms of v_1, v_2, and i_1 are, respectively, $V_1(s)$, $V_2(s)$, and $I_1(s)$, the $\mathfrak{L}$-transform equations for nodes 1 and 2 are

$$\left.\begin{aligned} &[(C_2 + C_3)s + G_1 + G_p]V_1(s) - C_3sV_2(s) = -I_1(s), \\ &-C_3sV_1(s) + [(C_1 + C_3)s + G_2]V_2(s) = 0. \end{aligned}\right\} \qquad [26]$$

Here $$I_1(s) = \mu G_pV_0(s) = \mu G_p\mathfrak{L}[v_0(t)].$$

The solution of equations 26 for $V_2(s)$ gives

$$V_2(s) = \frac{a_1sV_0(s)}{s^2 + b_1s + b_0}, \qquad [27]$$

in which

$$a_1 \triangleq -\frac{\mu G_pC_3}{C_1C_2 + C_2C_3 + C_1C_3} = -73.7 \times 10^6,$$

$$b_1 \triangleq \frac{G_2(C_2 + C_3) + (G_1 + G_p)(C_1 + C_3)}{C_1C_2 + C_2C_3 + C_1C_3} = 3.52 \times 10^5,$$

$$b_0 \triangleq \frac{G_2(G_1 + G_p)}{C_1C_2 + C_2C_3 + C_1C_3} = 9.88 \times 10^7.$$

The roots of the characteristic equation,

$$s^2 + b_1 s + b_0 = 0,$$

are

$$s_1 \triangleq -281, \quad \text{and} \quad s_2 \triangleq -3.52 \times 10^5. \qquad [28]$$

Each of the first $n - 1$ stages has a relation between output and input voltages similar to that expressed in equation 27 for the first stage. Considering now the nth stage, the $\mathfrak{L}$ transform equation for node $2n - 1$ is

$$(C_2 s + G_p + G_l)V_{2n-1}(s) = -I_n(s). \qquad [29]$$

Here

$$I_n(s) = \mu G_p V_{2n-2}(s) = \mu G_p \mathfrak{L}[v_{2n-2}(t)].$$

The solution for $V_{2n-1}(s)$ is

$$V_{2n-1}(s) = \frac{a_2}{s - s_3} V_{2n-2}(s), \qquad [30]$$

in which

$$a_2 \triangleq -\frac{\mu G_p}{C_2} = -11.7 \times 10^7, \quad \text{and} \quad s_3 \triangleq -\frac{G_p + G_l}{C_2} = -39.2 \times 10^4.$$

The input-output relations of the last stage, and of each successive preceding stage, form the equation set:

$$\left.\begin{aligned}
V_{2n-1}(s) &= \frac{a_2}{s - s_3} V_{2n-2}(s),\\
V_{2n-2}(s) &= \frac{a_1 s}{(s - s_1)(s - s_2)} V_{2n-4}(s),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
V_2(s) &= \frac{a_1 s}{(s - s_1)(s - s_2)} V_0(s).
\end{aligned}\right\} \qquad [31]$$

Solving the set 31 for $V_{2n-1}(s)$ in terms of $V_0(s)$, there is obtained

$$V_{2n-1}(s) = \frac{a_2}{s - s_3}\left[\frac{a_1 s}{(s - s_1)(s - s_2)}\right]^{n-1} V_0(s), \qquad [32]$$

which is the transform equation for the output voltage of an n-stage, RC-coupled amplifier in terms of the input voltage, all stages except the nth being alike.

If $n = 3$, and $v_0(t) \triangleq u(t)$, then $V_0(s) = s^{-1}$, and equation 32 becomes

$$V_5(s) = \frac{a_1^2 a_2 s}{(s - s_1)^2(s - s_2)^2(s - s_3)}. \qquad [33]$$

The $\mathfrak{L}^{-1}$ transformation of equation 33 gives, in accordance with the principles presented in Sec. 6, Chapter 6,

$$v_5(t) \;(=)\; (K_{11}t + K_{12})e^{s_1t} + (K_{21}t + K_{22})e^{s_2t} + K_3e^{s_3t}, \quad 0 \leq t, \qquad [34]$$

in which

$$K_{11} \triangleq \Big[(s - s_1)^2V_5(s)\Big]_{s=s_1} = \frac{a_1^2a_2s_1}{(s_1 - s_2)^2(s_1 - s_3)} = 3.67 \times 10^9,$$

$$K_{12} \triangleq \left[\frac{d}{ds}(s - s_1)^2V_5(s)\right]_{s=s_1} = a_1^2a_2\frac{(s_1 + s_2)s_3 - 2s_1^2}{(s_1 - s_2)^3(s_1 - s_3)^2}$$

$$= -1.30 \times 10^7,$$

$$K_{21} \triangleq \Big[(s - s_2)^2V_5(s)\Big]_{s=s_2} = \frac{a_1^2a_2s_2}{(s_2 - s_1)^2(s_2 - s_3)} = 4.50 \times 10^{13},$$

$$K_{22} \triangleq \left[\frac{d}{ds}(s - s_2)^2V_5(s)\right]_{s=s_2} = a_1^2a_2\frac{(s_1 + s_2)s_3 - 2s_2^2}{(s_2 - s_1)^3(s_2 - s_3)^2}$$

$$= -9.97 \times 10^8$$

$$K_3 \triangleq \Big[(s - s_3)V_5(s)\Big]_{s=s_3} = \frac{a_1^2a_2s_3}{(s_3 - s_1)^2(s_3 - s_2)^2} = 1.01 \times 10^9.$$

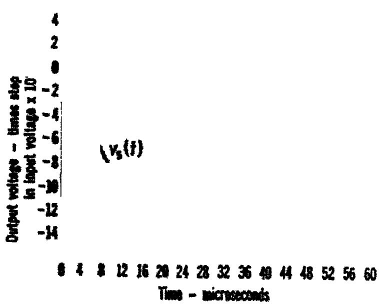


FIG. 7·11. Response of three-stage GC-coupled amplifier to unit step change of input-grid voltage.

Finally, with t in microseconds, the output voltage is

$$v_5(t) \;(=)\; [(3.67 \times 10^{-3}t - 13.0)e^{-2.81\times10^{-3}t}$$
$$+ (45.0t - 997)e^{-0.362t} + 1.01 \times 10^3e^{-0.392t}]\,10^6 \text{ volts}.$$

Its form is given in Fig. 7·11.

7. FILM HEAD WITH MECHANICAL FILTER

To show the application of the $\mathfrak{L}$ transformation to the solution of a mechanical-system problem, a mechanical filter will be used for an example.

For high-fidelity talking motion-picture performance the film speed past the scanning light must remain as nearly constant as possible, otherwise a frequency modulation of the reproduced sound results. In Fig. 7·12 is shown a form of mechanical filter [Co 2] that will filter out variations in film speed caused by irregularities in the driving gear trains and in the engagement of the sprocket teeth with the holes in the film.

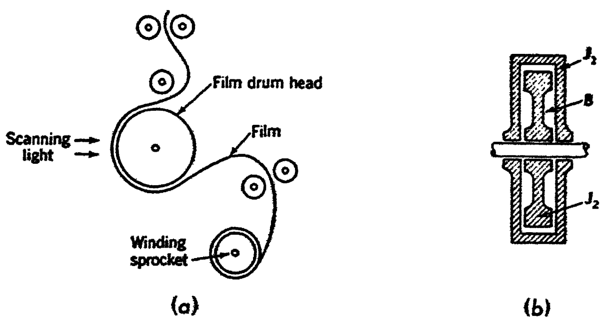


FIG. 7·12. A rotational-mechanical filter used in a motion-picture projector.

$$J_1 = 1.84 \times 10^4 \text{ dyne cm sec}^2 \text{ per radian.}$$
$$J_2 = 8.43 \times 10^4 \text{ dyne cm sec}^2 \text{ per radian.}$$
$$B = 12.4 \times 10^4 \text{ dyne cm sec per radian.}$$
$$K = 2.89 \times 10^6 \text{ dyne cm per radian.}$$
$$r = \frac{\text{sprocket diameter}}{\text{drum diameter}} = 0.578.$$

The film head is a hollow drum of small moment of inertia J_1. Within it is a concentric inner flywheel of moment of inertia J_2 which is large compared with J_1. The remainder of the space within the drum is filled with oil. The inner flywheel rotates on precision ball bearings on the drum shaft. The only coupling between drum and flywheel is through fluid friction and the very small friction in the ball bearings. Assuming that the frictional torque between the two flywheels is proportional to the difference in their angular velocities, the coupling between the flywheels can be accounted for by the rotational resistance B. The necessary spring restoring force for the filter system is provided by the flection of the film loops between drum head and idler pulleys when the film is running rapidly through the system. This spring effect is accounted for by the rotational stiffness K. The diameter of the winding sprocket that controls the linear velocity of the film at its pulling end is r times the drum diameter.

With the winding sprocket, the drum, and the flywheel running at their normal uniform angular velocities, the winding sprocket introduces a disturbance equivalent to a unit increase in its angular velocity for 0.15 second followed by a resumption of its normal velocity. Assuming that the film in contact with the drum cannot slip, find the change in angular velocity of the drum as a result of this pulse in the angular velocity of the winding sprocket.

This filter is a two-coordinate system, the two dependent variables being the changes in the angular velocities of drum and flywheel resulting from the change in angular velocity of the sprocket. Let $t = 0$ when the increase in sprocket angular velocity occurs, and let

$\omega_0(t)$ be the deviation of the sprocket from its normal angular velocity (this deviation is a known velocity pulse of unit amplitude and 0.15 second duration),
$\omega_1(t)$ be the deviation of the drum from its normal angular velocity,
$\omega_2(t)$ be the deviation of the inner flywheel from its normal angular velocity.

Since at $t = 0$ the sprocket, the drum, and the flywheel are running uniformly at their normal angular velocities, the initial conditions are equivalent to those of rest, i.e.,

$$\omega_1(0) = 0, \qquad \omega_0^{(-1)}(0) = 0,$$
$$\omega_2(0) = 0, \qquad \omega_1^{(-1)}(0) = 0.$$

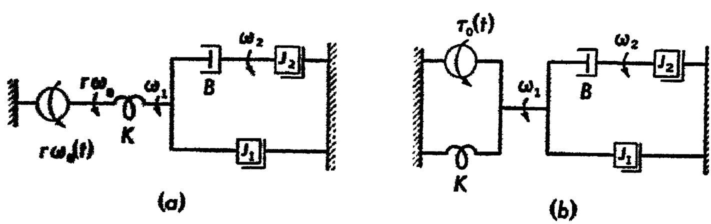


FIG. 7·13. Network diagram of the mechanical system shown in Fig. 7·12.

In Fig. 7·13–*a* the mechanical-network diagram of the system is shown with a velocity source $r\omega_0(t)$ in series with the rotational stiffness K. For convenience in analysis, this series branch can be replaced by a torque source $\tau_0(t)$ in parallel with K as in Fig. 7·13–*b*. With the aid of Table 2, Chapter 2, $\tau_0(t) = rK\int_0^t \omega_0(t)dt$, the additive constant being 0 since $\omega_0^{(-1)}(0) = 0$.

By inspection of the network diagram (Fig. 7·13-b) the i-d equations for the system can be written directly as

$$\left.\begin{aligned} J_1\frac{d\omega_1}{dt} + B\omega_1 + K\int_0^t \omega_1 dt - B\omega_2 &= \tau_0(t), \\ -B\omega_1 + J_2\frac{d\omega_2}{dt} + B\omega_2 &= 0. \end{aligned}\right\} \qquad [35]$$

If preferred, the equations can be written from inspection of the mechanical system itself (Fig. 7·12),

$$\left.\begin{aligned} J_1\frac{d\omega_1}{dt} &= K\int_0^t (r\omega_0 - \omega_1)dt + B(\omega_2 - \omega_1), \\ J_2\frac{d\omega_2}{dt} &= B(\omega_1 - \omega_2). \end{aligned}\right\} \qquad [36]$$

In equations 35 and 36 the presence of the definite integrals indicates that the conditions on the initial angles of sprocket and drum have been accounted for.

If the $\mathfrak{L}$ transforms of $\tau_0(t)$, $\omega_1(t)$, and $\omega_2(t)$ are $\mathrm{T}_0(s)$, $\Omega_1(s)$, and $\Omega_2(s)$, the $\mathfrak{L}$ transformation of equations 35 yields the equations

$$\left.\begin{aligned} \left(J_1 s + B + \frac{K}{s}\right)\Omega_1(s) - B\Omega_2(s) &= \mathrm{T}_0(s), \\ -B\Omega_1(s) + (J_2 s + B)\Omega_2(s) &= 0. \end{aligned}\right\} \qquad [37]$$

Here

$$\mathrm{T}_0(s) = \frac{rK}{s}\,\Omega_0(s) = rK\mathfrak{L}\left[\int_0^t \omega_0(t)dt\right].$$

The solution of equations 37 for $\Omega_1(s)$ is

$$\Omega_1(s) = \frac{rK}{J_1}\cdot\frac{(s + a_0)\Omega_0(s)}{s^3 + b_2 s^2 + b_1 s + b_0}, \qquad [38]$$

in which

$$\frac{rK}{J_1} = 90.8, \qquad b_1 \triangleq \frac{K}{J_1} = 1.57 \times 10^2,$$

$$a_0 \triangleq \frac{B}{J_2} = 1.47; \qquad b_0 \triangleq \frac{BK}{J_1 J_2} = 2.31 \times 10^2.$$

$$b_2 \triangleq \frac{B(J_1 + J_2)}{J_1 J_2} = 8.20,$$

The characteristic equation,

$$s^3 + b_2 s^2 + b_1 s + b_0 = 0,$$

has the roots $s_1 \triangleq -1.58$ and $s_2, s_3 \triangleq -3.32 \pm j11.6$.

The sprocket angular-velocity deviation $\omega_0(t)$ is a pulse of unit amplitude and 0.15 second duration. It can be expressed as the difference of two unit step functions, the second one of negative sign and displaced 0.15 second from the first, as

$$\omega_0(t) \triangleq u(t) - u(t - 0.15). \qquad [39]$$

Its $\mathfrak{L}$ transform is

$$\Omega_0(s) = \frac{1}{s} - \frac{1}{s} e^{-0.15s}, \qquad [40]$$

in which use has been made of pair 13, Table 1, Chapter 4. Thus equation 38 can be written

$$\begin{aligned} \Omega_1(s) &= \frac{rK}{J_1} \frac{s + a_0}{s(s - s_1)(s - s_2)(s - s_3)} (1 - e^{-0.15s}) \\ &= \tilde{\Omega}_1(s)(1 - e^{-0.15s}), \end{aligned} \qquad [41]$$

in which

$$\tilde{\Omega}_1(s) \triangleq \frac{rK}{J_1} \cdot \frac{s + a_0}{s(s - s_1)(s - s_2)(s - s_3)}.$$

The $\mathfrak{L}^{-1}$ transformation of equation 41 is the difference of two functions, the second one being equal in magnitude but opposite in sign to the first and displaced from the first by 0.15 second. Using the principles presented in Sec. 3, Chapter 6,

$$\begin{aligned} \omega_1(t)\ (=)\ & K_0 + K_1 e^{s_1 t} + \mathcal{R}[2K_2 e^{s_2 t}] \\ & -\{K_0 + K_1 e^{s_1 \hat{t}} + \mathcal{R}[2K_2 e^{s_2 \hat{t}}]\} u(\hat{t}), \qquad 0 \leqq t, \end{aligned} \qquad [42]$$

in which $\hat{t} \triangleq t - 0.15$, and

$$K_0 \triangleq \Big[s\tilde{\Omega}_1(s) \Big]_{s=0} = \frac{rK}{J_1} \cdot \frac{a_0}{b_0} = 0.578,$$

$$K_1 \triangleq \Big[(s - s_1)\tilde{\Omega}_1(s) \Big]_{s=s_1} = \frac{rK}{J_1} \cdot \frac{s_1 + a_0}{s_1(s_1 - s_2)(s_1 - s_3)} = 0.046,$$

$$K_2 \triangleq \Big[(s - s_2)\tilde{\Omega}_1(s) \Big]_{s=s_2} = \frac{rK}{J_1} \cdot \frac{s_2 + a_0}{s_2(s_2 - s_1)(s_2 - s_3)} = 0.326 e^{j165°}.$$

For $0 \leq t$, the drum's angular velocity in radians per second changes as follows:

$$\omega_1(t) = 0.578 + 0.046e^{-1.58t} + 0.652e^{-3.32t}\cos(11.6t + 165°)$$
$$- [0.578 + 0.046e^{-1.58\hat{t}} + 0.652e^{-3.32\hat{t}}\cos(11.6\hat{t} + 165°)]u(\hat{t}), \quad [43]$$

in which $\hat{t} \triangleq (t - 0.15)$ seconds. The form of $\omega_1(t)$ is shown in Fig. 7·14. A different treatment of the type of transform occurring in equation 41 is suggested in Chapter 8, problem 8·11.

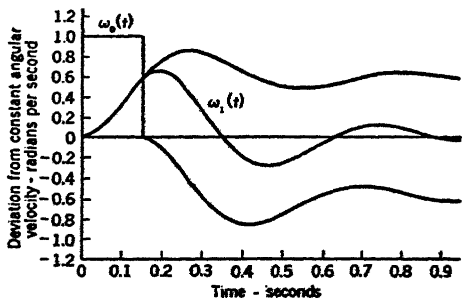


FIG. 7·14. Response of the film head to rectangular pulse in angular velocity of the winding sprocket.

8. AUTOMATIC CONTROL THROUGH TRANSIENT FEEDBACK

Automatic control [HA 10, IV 1, MA 7, MI 3, WE 3] through transient feedback is used to regulate a continuous process so that a specified variable quantity may be maintained at a desired or normal value. This normal may be either a constant or an independently varying function. If, as a result of extraneous disturbances, the variable deviates from this normal value, the ensuing transient is used to make the system self-correcting, thereby restoring the variable to the normal value. The process or system which is controlled is called the *controlled system*. The combination of devices that continuously measures the deviation of the variable and translates this deviation into a correction of system input is called the *automatic controller* or, briefly, controller. In measuring deviations of the variable, a good controller extracts negligible energy from the output of the controlled system, but the corrective energy which it regulates at the input of the controlled system may be large. Automatic controllers are used, for example, in regulating temperature, liquid level, pressure, fluid flow, voltage, frequency, speed, angular position, sound-volume level, and in stabilizing the motion of ships, in steering ships, and in piloting aircraft. Closely allied is the control of

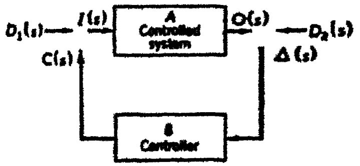


FIG. 7·15. Schematic diagram of a system having automatic control.

amplification and waveform in vacuum-tube amplifiers through feedback [BL 1].

There are certain relations, fundamental to automatic-control systems, which can be formulated concisely and conveniently by use of transform equations since these equations are algebraic in nature. A statement of these relations can be made by reference to the schematic diagram of Fig. 7·15. Using $\mathfrak{L}T$ as an abbreviation for "$\mathfrak{L}$ transform," the symbols in Fig. 7·15 have the following meaning:

$D_1(s)$ is the $\mathfrak{L}T$ of the local disturbance $d_1(t)$ at input end of A,
$C(s)$ is the $\mathfrak{L}T$ of the correction $c(t)$ at input end of A,
$I(s)$ is the $\mathfrak{L}T$ of the corrected input disturbance $i(t)$ entering A,
$O(s)$ is the $\mathfrak{L}T$ of the resulting output disturbance $o(t)$ leaving A,
$D_2(s)$ is the $\mathfrak{L}T$ of the local disturbance $d_2(t)$ at output end of A,
$\Delta(s)$ is the $\mathfrak{L}T$ of the total deviation $\delta(t)$ at output end of A.

Let

$G_{12}(s)$ be the transfer function for A, usually denoted by μ in amplifier theory,
$G_{34}(s)$ be the transfer function for B, sometimes denoted by β.

Then if (1) the $\lim_{s\to\infty} G_{12}(s) < \infty$, and the $\lim_{s\to\infty} G_{34}(s) < \infty$, and (2) it is assumed that the system has been operating without a deviation for sufficient time for all effects of previous deviations to have disappeared, the fundamental equations for automatic control can be written

$$C(s) = G_{34}(s)\Delta(s), \tag{44}$$

$$I(s) = D_1(s) + C(s), \tag{45}$$

$$O(s) = G_{12}(s)I(s), \tag{46}$$

$$\Delta(s) = O(s) + D_2(s). \tag{47}$$

By placing the above restrictions on the transfer functions and on the operating condition, the equations 44 to 47 can be written without concern for the initial values of the disturbances and their derivatives except that they should be finite, and the initial values within the system can be taken as zero.

When the problem is such that the restrictions specified above are not met, the $\mathfrak{L}$ transformation of the i-d equations of the controlled

system and the controller, introducing therewith the necessary initial conditions, will provide a set of algebraic equations which will serve in place of equations 44 to 47. No attempt will be made here, however, to present a set sufficiently general to cover all possible combinations. It is sufficient to remark that the $\mathfrak{L}$-transformation method presents a mechanism by means of which the explicit problem can be expressed algebraically.

Substituting from equation 44 in 45, from equation 45 in 46, and from 46 in 47 and solving for $\Delta(s)$, there is obtained

$$\Delta(s) = \frac{G_{12}(s)}{1 - G_{12}(s)G_{34}(s)} D_1(s) + \frac{1}{1 - G_{12}(s)G_{34}(s)} D_2(s). \quad [48]$$

The common factor $[1 - G_{12}(s)G_{34}(s)]^{-1}$ can be called the *control factor* for the entire system. Let it be represented by $G(s)$. Then

$$\Delta(s) = G(s)[G_{12}(s)D_1(s) + D_2(s)]. \quad [49]$$

In the design of a controller for a given system, the transfer function $G_{12}(s)$ is a characteristic of that system and usually cannot be changed unless the system is changed. If a change in the system is not feasible, only $G_{34}(s)$ is subject to choice to yield a most favorable control factor $G(s)$.[1] The locations in the s-plane of the zeros and singularities of $G(s)$ are of great importance since they determine the character of the response. Evidently the overall system should have no poles in the right half-plane, and no multiple-order poles on the axis of imaginaries, since these yield terms in the response that grow without bound [Ro 2, Ny 1]. Furthermore, there should be no first-order conjugate poles on the axis of imaginaries since these yield terms in the response that oscillate with constant amplitude. In other words, for automatic control the type and distribution of poles should be such as to provide a stable and nonoscillating system. On the other hand, it is ordinarily desirable for $G(s)$ to have a zero at the origin of sufficiently high order to cancel a pole of given order at this point likely to be introduced by the function $[G_{12}(s)D_1(s) + D_2(s)]$. The elimination of a pole at the origin insures the absence of a constant deviation or a deviation that increases as a power of t. Since the design of controllers is outside the field of this text, these properties of $G(s)$ will not be discussed further here.

Equation 49 is of great generality, applying equally as well to systems with distributed constants as to systems with lumped constants. Since

[1] J. Taplin in 1937 in an unpublished memorandum used the control factor $[1 - G_{12}(s)G_{34}(s)]^{-1}$ to summarize the theory of design of an automatic-control system, carrying over the basic idea from electric-amplifier design [Bl 1, Ny 1].

the latter are our concern at this point, a lumped-constant system will be used for an illustrative example.

In Fig. 7·16 is shown a simple servo-mechanism [HA 10], a form of automatic-control system in which the disturbance may be considered to be at the output end of the controlled system. The servomotor and its shaft load have a moment of inertia J. The damping is represented by the rotational resistance B. It is desired to have the angular position $\phi_1(t)$ of the motor shaft coincide with the angular position $\phi_2(t)$ of the shaft shown on the right. The difference $\phi_1(t) - \phi_2(t)$ between their angular positions is the deviation $\phi(t)$ and actuates the controller which in turn gives to the motor shaft a correction torque $\tau_c(t)$. Assume that the controller equation is

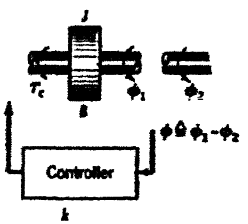


FIG. 7·16. A simple servo-mechanism with proportional-plus-integrating control.

$J = 60$ $\quad k_1 = 12 \times 10^4$
$B = 4.5 \times 10^3$ $\quad k_2 = 11 \times 10^5$

$$\tau_c(t) = -k_1\phi(t) - k_2\int\phi(t)dt, \tag{50}$$

in which k_1 and k_2 are positive real constants. This equation states that the instantaneous correction torque is dependent upon both the instantaneous deviation and the integrated deviation up to that instant. Furthermore, the minus signs show that when the deviation and its time integral are positive, the correction torque is a retarding torque. When the control equation is of the form given in equation 50 the controlled system is said to have proportional-plus-integrating control.

Assuming that the values of the servo-mechanism constants, in any single system of units, are as given in Fig. 7·16, find the resulting deviation $\phi(t)$ if the reference-shaft angle $\phi_2(t)$ suddenly starts to increase linearly with time. Assume that at the start of this disturbance the time integral of the deviation is zero.

The deviation $\phi(t)$ and the angle $\phi_1(t)$ are chosen as the dependent variables. The angle $\phi_2(t)$ of the reference shaft has been chosen here as $tu(t)$. The differential and integral equations of the system are

$$J\frac{d^2\phi_1}{dt^2} = -B\frac{d\phi_1}{dt} + \tau_c(t), \tag{51}$$

$$\tau_c(t) = -k_1\phi - k_2\int\phi dt, \tag{52}$$

in which

$$\phi \triangleq \phi_1 - \phi_2 = \phi_1 - tu(t). \tag{53}$$

The initial conditions are

$$\phi_1(0) = 0, \qquad \phi_1'(0) = 0, \qquad \phi^{(-1)}(0) = 0.$$

Eliminating $\tau_c(t)$ from equations 51 and 52, and using the value of $\phi^{(-1)}(0)$, there is obtained

$$J\frac{d^2\phi_1}{dt^2} + B\frac{d\phi_1}{dt} = -k_1\phi - k_2\int_0^t \phi dt. \tag{54}$$

Let the $\mathfrak{L}$ transforms of the functions ϕ and ϕ_1 of t be denoted by the functions Φ and Φ_1 of s. Then the $\mathfrak{L}$ transformation of equations 53 and 54 yields

$$\left.\begin{aligned} \Phi(s) &= \Phi_1(s) - \frac{1}{s^2}, \\ (Js^2 + Bs)\Phi_1(s) &= -\left(k_1 + \frac{k_2}{s}\right)\Phi(s). \end{aligned}\right\} \tag{55}$$

Solution of equations 55 for $\Phi(s)$ gives

$$\Phi(s) = -\frac{s + a_0}{s^3 + b_2s^2 + b_1s + b_0}, \tag{56}$$

in which

$$\begin{aligned} a_0 &\triangleq B/J = 75 \\ b_2 &\triangleq B/J = 75 \\ b_1 &\triangleq k_1/J = 2.0 \times 10^3 \\ b_0 &\triangleq k_2/J = 1.83 \times 10^4. \end{aligned}$$

Equation 56 is an example of the general relation 49 in which

$$\begin{aligned} \Delta(s) &= \Phi(s), \\ G_{12}(s) &= \frac{1}{Js^2 + Bs}, \\ G_{34}(s) &= -\left(k_1 + \frac{k_2}{s}\right), \\ G(s) &= \frac{(s + a_0)s^2}{s^3 + b_2s^2 + b_1s + b_0}, \\ D_1(s) &= 0, \\ D_2(s) &= -s^{-2}. \end{aligned}$$

It will be observed that it is the presence of k_2 in the equation for the correction torque that gives $G(s)$ the zero of second order at the origin and eliminates the pole that would be there due to $D_2(s)$. This is seen in equation 56. If k_2 were zero, then b_0 would be zero, and there would be a pole at the origin, which would produce a constant deviation.

The characteristic equation,

$$s^3 + b_2 s^2 + b_1 s + b_0 = 0,$$

has the roots $s_1 \triangleq -21.6$, and $s_2, s_3 \triangleq -26.7 \pm j11.5$.

The $\mathfrak{L}^{-1}$ transformation of equation 56 gives, by application of the methods developed in Sec. 3, Chapter 6,

$$\phi(t) \;(=)\; \mathfrak{L}^{-1}\left[-\frac{s + a_0}{(s - s_1)(s - s_2)(s - s_3)}\right]$$

$$= K_1 e^{s_1 t} + \mathcal{I}[2jK_2 e^{s_2 t}], \qquad 0 \leq t, \qquad [57]$$

in which

$$K_1 \triangleq [(s - s_1)\Phi(s)]_{s=s_1} = -\frac{s_1 + a_0}{(s_1 - s_2)(s_1 - s_3)} = -0.338,$$

$$K_2 \triangleq [(s - s_2)\Phi(s)]_{s=s_2} = -\frac{s_2 + a_0}{(s_2 - s_1)(s_2 - s_3)} = -\frac{0.343}{2j}\, e^{-j101°}.$$

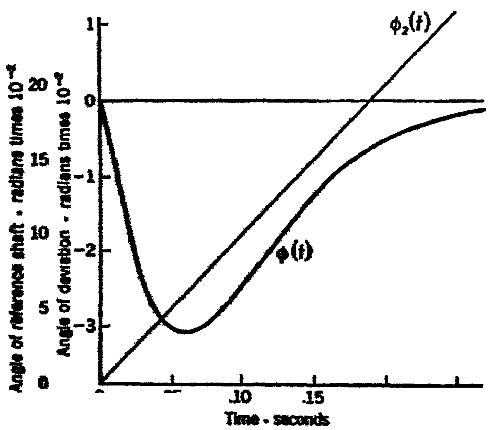


FIG. 7·17. Lag of servo-driven shaft when angle of reference shaft increases linearly.

The deviation resulting from the linear increase of ϕ_2 is thus

$$\phi(t) \;(=)\; -0.338e^{-21.6t} - 0.343e^{-26.7t} \sin(11.5t - 101°), \quad 0 \leq t. \qquad [58]$$

Graphs of this deviation and the disturbance that causes it are given in Fig. 7·17. Beginning at $t = 0$, ϕ_1 falls behind ϕ_2 but later catches up with it.

9. Conditions for Stability; Conditions for Undamped Oscillations

A more general problem in the theory of automatic control is the determination of limiting relations that must hold among the constants of the controller and the controlled system if the overall system is to be stable and nonoscillatory in the steady state. On the other hand, if sustained oscillations are desired, as for example in an oscillator, there is the problem of finding the exact relations among the constants which will permit such oscillations to exist [GI 1, MA 6].

The overall system will be stable and nonoscillatory in the steady state if all the roots of the characteristic equation have negative real parts [Ro 2]. In fact it will have these properties even if the equation has a single zero root. It will be stable but oscillatory if there are conjugate imaginary roots all different. It will be unstable if there are roots with positive real parts, or if there are repeated zero or conjugate imaginary roots.

If the characteristic equation has been solved and its roots are known, one may easily see whether or not the system is stable. But equations of third and higher degree usually are not solved without considerable labor. It is desirable therefore to be able to settle the question of stability without actually solving the characteristic equation. There is a method due to Routh [Ro 2] by which the number of roots present having positive real parts can be determined from the coefficients of this equation without actually finding the roots. The method gives also the number of roots having zero real parts. This method is an extension to complex roots of Budan's theorem [DI 1, 83–85] and Descartes' rule of signs. A summary of the method, useful for purposes here, is given below without proof.

Stating this method in our notation, let the characteristic equation, from which any zero roots have been removed, be represented by

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0, \qquad [59]$$

in which the b's are real coefficients, with b_n positive, and n is a positive real integer. Arrange the coefficients in two rows as

$$\begin{array}{l|llll} s^n & b_n & b_{n-2} & b_{n-4} & \cdots \\ s^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \end{array}$$

The coefficients of a third row are found by cross-multiplication as follows:

$$\begin{array}{l|lll} s^{n-2} & \dfrac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}}, & \dfrac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}, & \ldots \end{array}$$

Using rows two and three, the coefficients of a fourth row are found by cross-multiplying again. New rows are formed in this way until no term remains. In the course of this development the coefficients in any row may be multiplied or divided by a positive number without altering the result. This simplifies the numerical work of finding the coefficients of the succeeding row. If all the terms in the first column are of one sign, the equation has no roots with positive real parts; if there are changes in sign, the number of changes equals the number of roots with positive real parts.

For example, let the characteristic equation be

$$(s+4)(s-2+j3)(s-2-j3)(s+1+j2)(s+1-j2)$$
$$= s^5 + 2s^4 + 2s^3 + 46s^2 + 89s + 260 = 0. \quad [60]$$

Following the procedure presented above, the table of coefficients is

	1	2	89	
s^4	2	46	260	
s^3	-1	-1.95		(after dividing by 21)
s^2	1	6.18		(after dividing by 42.1)
s^1	4.23			
s^0	6.18			

In the first column there are two changes of sign, indicating that the specified characteristic equation has two roots with positive real parts — and this is seen to be correct, for there are roots at $2 + j3$ and $2 - j3$. Note that the method does not actually locate the roots but only indicates whether there are roots present with positive real parts, and if there are, tells how many. There are two exceptions to this general process which need special comment.

1. An exception to the general process arises when the *first-column* term in any row is zero, but the remaining terms in this row are not all zero. Any attempt to write the following row fails because its terms will be infinite. Three methods of overcoming this difficulty are:

a. Replace the zero by an arbitrarily small real number ϵ and proceed as usual. Terms in ϵ^2 will need to be retained only if there is uncertainty as to the relative magnitudes of derived coefficients. The total number of sign changes in the first column will be the same whether ϵ is considered positive or negative.

b. Multiply the equation under investigation by a factor such as $(s + h)$, with h a positive real number. This will increase the degree of the equation but will restore the missing power of s. To avoid producing a case coming under the second exception, treated below, $-h$ should not be a root of the original equation.

c. Form the equation whose roots are the reciprocals of those of the

original equation. This can be done by replacing s by u^{-1} and clearing of negative powers of u. The resulting equation in u has the coefficients of the original equation in the reverse order. Use is made here of the facts that (1) the reciprocal of a root is a number whose real part has the same sign as the real part of the root and (2) the reciprocal of an imaginary root is an imaginary number.

2. A second exception to the general process arises if *all* the coefficients in the second or any derived row are zero. If this occurs there are present roots of equal order lying radially opposite each other and equidistant from the origin. The process can be continued by writing in place of the row of zeros the coefficients of the derivative of an auxiliary polynomial whose coefficients are the numbers in the last nonvanishing row. In this auxiliary polynomial s will appear only in even powers, the highest power being that of the s indicated at the left of the last nonvanishing row. The roots of the equation formed by equating the auxiliary polynomial to zero are all roots of the original equation. They occur in pairs, the two roots of a pair being equal in magnitude and opposite in sign. As before, the variations of sign in the first column of the coefficient table will give the number of roots having positive real parts. The remaining roots will have negative or zero real parts. If there are any roots with zero real parts they will be found among the roots of the auxiliary equation.

To illustrate this special procedure consider the characteristic equation,

$$(s + j2)(s - j2)(s + 1)(s - 1)(s + 3)$$
$$= s^5 + 3s^4 + 3s^3 + 9s^2 - 4s - 12 = 0. \quad [61]$$

On forming the rows,

s^5	1	3	-4	
s^4	1	3	-4	(after dividing by 3)

it is seen that the third row will be composed of zeros. The occurrence of two identical rows indicates the presence of roots equal in absolute magnitude but lying radially opposite. The auxiliary polynomial $s^4 + 3s^2 - 4$ is formed using the second-row coefficients, since the second is the last nonvanishing row. The coefficients of the derivative of this polynomial (after dividing by 4) are taken for the third row, and the process continued. The third and the remaining rows are thus

s^3	1	1.5
s^2	1.5	-4
s^1	4.17	
s^0	-4	

There is one change of sign in the first column, indicating that there is one root with positive real part. The remaining roots have either negative real parts or zero real parts. If there are any with zero real parts, they will be found among the roots of the auxiliary equation,

$$s^4 + 3s^2 - 4 = 0. \qquad [62]$$

These four roots are ± 1 and $\pm j2$, and all are roots of the original equation. They are the pairs of roots that produced the two identical rows.

Returning now to the general question of stability, it is seen from the above discussion that there will be no roots with positive real parts, and the system will be stable and nonoscillatory in the steady state, if all the coefficients of the characteristic equation,

$$b_n s^n + b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0 = 0, \qquad [59]$$

are positive, and

$$b_n b_{n-3} < b_{n-1} b_{n-2},$$
$$b_n b_{n-3}^2 + b_{n-1}^2 b_{n-4} < b_{n-1} b_{n-2} b_{n-3} + b_n b_{n-1} b_{n-5},$$

for all the derived coefficients that will fall in the first column.

If sustained undamped oscillations are desired, the coefficients must be such as to (1) make all the coefficients of one row of the coefficient table zero and (2) cause the auxiliary polynomial formed from the coefficients of the last nonvanishing row to have a pair of conjugate imaginary zeros. Like the criterion for stability above, this criterion for sustained oscillations becomes too complicated for statement explicitly in terms of the coefficients of the general polynomial.

The simple servo-mechanism which was considered in the previous section will be used as an example, except that here its constants will be taken as unspecified numerically. The characteristic equation for this mechanism was found to be of the form

$$s^3 + b_2 s^2 + b_1 s + b_0 = 0. \qquad [63]$$

The coefficient table is

$$\begin{array}{ll} 1 & b_1 \\ b_2 & b_0 \\ \dfrac{b_2 b_1 - b_0}{b_2} & \\ b_0 & \end{array}$$

If there are to be no roots with positive real parts, b_2, b_1, and b_0 must be positive and $b_0 < b_2 \cdot b_1$. In terms of the constants of the system, this means that k_1 and k_2 must be positive, and $J/B < k_1/k_2$.

An undamped oscillation can exist if the third row in the table is zero, i.e., if $b_0 = b_1 \cdot b_2$. For this condition, the auxiliary equation is

$$b_2 s^2 + b_0 = 0, \tag{64}$$

and its roots are $\pm j\sqrt{b_0/b_2}$. In terms of the constants of the system, this oscillation will occur if $k_1/k_2 = J/B$, and its angular frequency will be $\sqrt{k_2/B}$.

10. Use of Increments as Dependent Variables

In solving a problem by means of the $\mathfrak{L}$ transformation the initial values of the dependent variables and their derivatives are introduced into the equations in the process of the transformation. But the determination of these initial values may be an irksome task which one would gladly avoid if possible. Frequently it can be avoided if a favorable choice of variables is made. Such a favorable choice is possible when a system is suddenly modified and only the subsequent *changes* or *increments* in the system dependent variables are desired. With the differential equations of the modified system written with the increments as dependent variables, the initial conditions for these increments are all zero.

This principle finds wide application in handling problems in automatic control where in general only deviations or increments are of interest. But here it will be presented in more generality in terms of an electric network suddenly modified by switching. Here the term " switching " includes accidental short-circuits and open-circuits as well as modifications made intentionally.

Since the networks considered are linear the principle of superposition applies. Each branch current (and voltage) in the network after switching may be considered to be composed of two components: (1) The branch current (or voltage) that would result if the network, with zero energy storage in its elements, were suddenly excited by a single source located at the point of switching. This single source is equivalent either to the voltage drop annulled by closing the switch, or to the current annulled by opening the switch. (2) The branch current (or voltage) that would exist if no switching took place.

The first component is the increment produced by the switching. When added to the second component it gives the total branch current (or voltage) existing after the switching.

The second component is zero when the variable sought is the current in the switch that has just closed, or is the voltage drop across the switch that has just opened. For these cases the first component con-

stitutes the entire solution. Since the initial conditions used in calculating the first component are all zero, the formulation of the necessary transform equations is especially easy. These two cases will be discussed in more detail in the following sections.

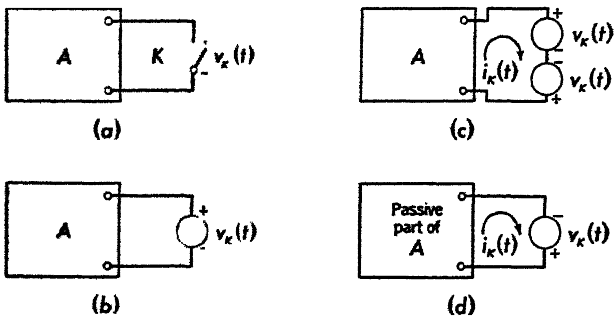


FIG. 7·18. Effects of closing a switch can be attributed to a voltage source inserted at point of switching.

11. CLOSING OF A SWITCH

In Fig. 7·18–*a* let A be a linear network two points of which are connected at $t = 0$ by closing the switch K. In general A will contain voltage and current sources. Let $v_K(t)$ be the voltage drop that is annulled by closing the switch, and let $i_K(t)$ be the current that results in the switch. If $V_K(s)$ and $I_K(s)$ are the $\mathfrak{L}$ transforms respectively of $v_K(t)$ and $i_K(t)$, and $Y_K(s)$ is the input-admittance function of A after all voltage sources have been short-circuited and all current sources have been open-circuited, and A is viewed from the terminals of K, then the relation between $I_K(s)$ and $V_K(s)$ can be shown to be

$$I_K(s) = Y_K(s)V_K(s). \qquad [65]$$

Furthermore, the $\mathfrak{L}$ transforms of the current and voltage-drop increments within A can be found from $I_K(s)$ by the use of transfer functions.

That equation 65 is valid may be shown by the following reasoning. If the open switch (Fig. 7·18–*b*) were replaced by a voltage source $v_K(t)$ with polarity the same as that of the voltage drop $v_K(t)$, conditions throughout A would remain unchanged. But if a second voltage source $v_K(t)$ were now connected (Fig. 7·18–*c*) in series opposition to the first, the voltage drop across the two sources would be zero, and conditions would be the same as with K closed. There would be a current in the branch connecting the two terminals of A and increments in the currents and voltage drops within A. All of these could be taken, by the principle of superposition, to be the consequence of impressing the second

voltage source on the passive part of A (Fig. 7·18–d) with all initial conditions zero.

Equation 65 is a transform equation that states a limiting case of a generalization of Thévenin's theorem [HE 4, TH 1, WI 6]. When written, this theorem pertained to the direct-current steady-state behavior of networks, but it has been generalized since to alternating currents and to transients. In the limiting form the external resistance is that of the closed switch, i.e., is zero.

The total currents and voltage drops in A following the annulment of the voltage drop across K can be found by adding to the increments, as calculated above, the values that would exist if K had not closed. One must be able to determine these second components without too great effort if this superposition method is to present any advantage over the usual straightforward procedure in which energy storage at the instant of switching is considered explicitly.

The following example is made simple so that the result found by use of equation 65 can be verified quickly by inspection.

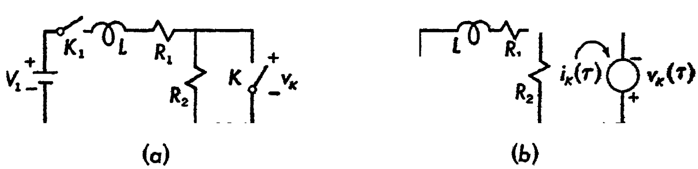


FIG. 7·19. The currents and voltage drops found in network b are the same as the *increments* in the corresponding variables in network a when switch K closes.

Example 1. In the circuit shown in Fig. 7·19–a, switch K_1 is closed at time $t = 0$, and t_1 seconds later switch K is closed. Let t_1 be less than the time constant $L(R_1 + R_2)^{-1}$ of the circuit, so that the transient is interrupted. The current in switch K is to be found.

Let $\tau \triangleq t - t_1$. For $0 \leqq \tau$, the voltage drop that is annulled by closing K is

$$v_K(\tau) = \frac{R_2 V_1}{R_1 + R_2}[1 - e^{-a(t_1+\tau)}], \qquad a \triangleq \frac{R_1 + R_2}{L}. \tag{66}$$

Denoting $\mathfrak{L}[v_K(\tau)]$ by $V_K(s)$, equation 66 when $\mathfrak{L}$ transformed is

$$V_K(s) = \frac{R_2 V_1}{R_1 + R_2}\left(\frac{1}{s} - \frac{e^{-at_1}}{s + a}\right) = \frac{R_2 V_1}{R_1 + R_2} \cdot \frac{(1 - e^{-at_1})s + a}{s(s + a)}. \tag{67}$$

For this network (Fig. 7·19–b),

$$Y_K(s) = \frac{1}{R_2} + \frac{1}{Ls + R_1} = \frac{1}{R_2} \cdot \frac{s + a}{s + b}, \qquad b \triangleq \frac{R_1}{L}. \tag{68}$$

Example 1. In the circuit shown in Fig. 7·21–*a* switch K_1 is closed at time $t = 0$, and t_1 seconds later switch K is opened. Let t_1 be less than the time constant R_1C of the circuit so that the transient is interrupted. The voltage drop across switch K after it opens is to be found.

FIG. 7·21. The currents and voltage drops found in network *b* are the same as the *increments* in the corresponding variables in network *a* when switch K opens.

Let $\tau \triangleq t - t_1$. For $0 \leqq \tau$, the current that is annulled by opening K is

$$i_K(\tau) = \frac{V_1}{R_1} e^{-a(t_1+\tau)}, \qquad a \triangleq \frac{1}{R_1C}. \tag{73}$$

Let the $\mathfrak{L}$ transform of $i_K(\tau)$ be denoted by $I_K(s)$; then the transformation of equation 73 gives

$$I_K(s) = \frac{V_1}{R_1} e^{-at_1} \frac{1}{s+a}. \tag{74}$$

For this network (Fig. 7·21–*b*),

$$Z_K(s) = \frac{R_2(R_1 + 1/Cs)}{R_1 + R_2 + 1/Cs} = \frac{R_1R_2}{R_1 + R_2} \cdot \frac{s+a}{s+b}, \qquad b \triangleq \frac{1}{(R_1 + R_2)C}.$$

Letting $v_K(\tau)$ be the voltage drop across the switch after it is opened, and letting $V_K(s) \triangleq \mathfrak{L}[v_K(\tau)]$, application of equation 72 gives

$$\begin{aligned} V_K(s) &= Z_K(s)I_K(s) \\ &= \frac{V_1R_2}{R_1 + R_2}\left(\frac{s+a}{s+b}\right)\left(\frac{e^{-at_1}}{s+a}\right) \\ &= \frac{V_1R_2}{R_1 + R_2} \frac{e^{-at_1}}{s+b}. \end{aligned} \tag{75}$$

The $\mathfrak{L}^{-1}$ transformation of equation 75 gives

$$\begin{aligned} v_K(\tau) &(=) \frac{V_1R_2}{R_1 + R_2} e^{-(at_1+b\tau)} \\ &= \frac{V_1 - V_1(1 - e^{-at_1})}{R_1 + R_2} \cdot R_2e^{-b\tau}, \qquad 0 \leqq \tau. \end{aligned} \tag{76}$$

Since $V_1(1 - e^{-at_1})$ is the voltage across the condenser at $\tau = 0$, it can be seen readily that the method has given the correct result.

13. Network Recovery Voltage

A practical application of the principle presented in Sec. 12 is made in the calculation of the recovery of voltage [Bo 2, Pa 3] across an oil circuit breaker in an a-c power network when this breaker interrupts a short-circuit current. When the breaker contacts separate, an arc forms which prolongs the life of the current beyond the time of physical separation of the contacts. This arc is extinguished a few cycles later at a zero value of the alternating current. After this interruption of current, the rate at which the recovery voltage across the breaker contacts is built up by the system is important. If the oil switch is to interrupt the short-circuit current successfully, the voltage breakdown strength of the space between its rapidly separating contacts must grow at a rate sufficiently great to withstand the voltage that the network is building up simultaneously across these same contacts. The problem then is to calculate this recovery voltage of the network.

For a simple example, assume that a single line-to-ground short circuit occurs on a three-phase transmission circuit on the line side of, but near, the current-limiting reactor (Fig. 7·22–*a*): The low-potential side of the transformer is connected to an "infinite bus," i.e., to a source whose power capacity is large compared with the power demands made by the circuits connected to it, and whose short-circuit impedance is small compared with the impedances of stationary apparatus in the network. If a circuit breaker located at the point shown is to open the faulted phase successfully, what rate of voltage build-up across its terminals must it withstand immediately after the current interruption?

The calculation of short-circuit currents in power networks under general unsymmetrical short-circuit conditions requires considerable background knowledge of symmetrical-component theory [Ly 1, Wa 3] and, in particular, facility in the use of the impedances to positive-, negative-, and zero-sequence currents in stationary and rotating electric apparatus and in transmission lines. This background knowledge will be assumed here because the calculation of the magnitude of the short-circuit current is incidental to the major problem of calculation of the recovery voltage.

Since the resistances in power circuits at the power frequency are small compared with the reactances, it is sufficient to consider only reactances in the calculation of the short-circuit current. As an aid in calculating the short-circuit current in the circuit breaker of Fig. 7·22–*a*, the circuit of Fig. 7·22–*b* is used. It consists of a series connection of the zero-, positive-, and negative-sequence networks to represent the single line-to-ground fault. For a transformer and a reactor the impedances to the three sequence currents are alike, and hence the three sequence networks

are alike except for the voltage of the infinite bus which is of positive sequence only. The transmission line and the load at its far terminal are assumed to have small effect upon the magnitude of the circuit-breaker current and are hence neglected.

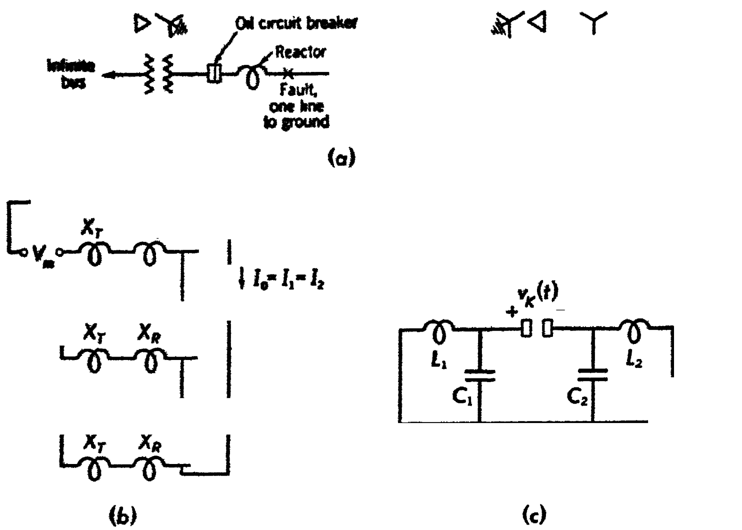


FIG. 7·22. (a) One-line diagram of a three-phase power circuit.
(b) Interconnection of the three sequence networks for a single line-to-ground fault.
(c) Network representing elements of faulted phase essential for calculation of recovery voltage across opened contacts of circuit breaker.

$L_1 = 0.018$ henry $C_1 = 10^{-3}$ microfarad $V_m = 38.1\sqrt{2}$ kv
$L_2 = 0.01$ henry $C_2 = 10^{-4}$ microfarad $\omega = 377$ radians per second

Let $v_n(t) = V_m \cos \omega t$ represent the voltage to neutral at the infinite bus, L_1 the transformer leakage inductance per phase, and L_2 the reactor inductance per line, then $x_T = \omega L_1$ is the transformer leakage reactance per phase, and $x_R = \omega L_2$ is the reactor reactance per line. If I_0, I_1, and I_2 are the magnitudes of the zero-, positive-, and negative-sequence currents, respectively, the magnitude of the short-circuit current in the circuit breaker is

$$I_m = I_0 + I_1 + I_2 = \frac{3V_m}{3(x_T + x_R)} = \frac{\sqrt{2} \times 38.1 \times 10^3}{2.8 \times 10^{-2} \times 377}$$

$$= 5.1 \times 10^3 \text{ amperes.} \qquad [77]$$

Then the current annulled by the opening of the circuit breaker is

$$i_K(t) = I_m \sin \omega t. \qquad [78]$$

In the calculation of the recovery voltage, interest centers principally on developments during the first fifty or so microseconds after the current interruption. The shunt capacitances to ground of the network elements play an important role in this brief interval and must be considered. In the network (Fig. 7·22–c) for the faulted phase, C_1 is the capacitance to ground of the transformer high-tension winding and terminal bushing and C_2 is the capacitance to ground of the reactor. Since it is the voltage built up by the system that is to be calculated, the capacitance to ground of the breaker is not included in the circuit. The capacitance to ground of the "infinite bus" is great compared with C_1 and C_2, and at the characteristic angular frequencies arising, it effectually short-circuits the small inductance of this source. The capacitance and inductance of this source are accordingly omitted.

Although the interrupted current, from equation 78, is sinusoidal only an interval of fifty or so microseconds after $t = 0$ is to be considered. Since this is an interval exceedingly small in comparison with the period, $2\pi/\omega = 16{,}700$ microseconds, of $i_K(t)$, it is sufficiently exact to take for $i_K(t)$ the first term $I_m\omega t$ of its power-series expansion. Hence $I_K(s) \approx \mathfrak{L}[I_m\omega t] = I_m\omega s^{-2}$.

The short-circuit input-impedance function as viewed from the circuit breaker is found by a series combination of two parallel-impedance functions and is

$$Z_K(s) = \frac{s}{C_1(s^2 + 1/L_1C_1)} + \frac{s}{C_2(s^2 + 1/L_2C_2)}.$$

Let $v_K(t)$ be the voltage drop across the open circuit breaker, and let $V_K(s)$ denote its $\mathfrak{L}$ transform. In accordance with equation 72,

$$V_K(s) = Z_K(s)I_K(s) = I_m\omega\left[\frac{s}{C_1(s^2 + \beta_1^2)} + \frac{s}{C_2(s^2 + \beta_2^2)}\right]\frac{1}{s^2}, \qquad [79]$$

in which $\beta_1 \triangleq \sqrt{1/L_1C_1} = 0.235 \times 10^6$, and $\beta_2 \triangleq \sqrt{1/L_2C_2} = 10^6$.

By use of the principles presented in Sec. 5, Chapter 6, the inverse transformation of equation 79 is found to be, for $0 \leqq t \ll 2\pi/\omega$,

$$v_K(t) \; (=) \; \mathfrak{L}^{-1}\left[\frac{I_m\omega}{C_1}\cdot\frac{1}{s(s^2 + \beta_1^2)} + \frac{I_m\omega}{C_2}\cdot\frac{1}{s(s^2 + \beta_2^2)}\right]$$

$$= K_1 + \Re[2K_2e^{j\beta_1 t}] + K_3 + \Re[2K_4e^{j\beta_2 t}], \qquad [80]$$

in which

$$K_1 \triangleq \left[\frac{I_m\omega}{C_1(s^2+\beta_1^2)}\right]_{s=0} = \frac{I_m\omega}{C_1\beta_1^2} = I_m\omega L_1 = 34.6 \times 10^3,$$

$$K_2 \triangleq \left[\frac{I_m\omega}{C_1 s(s+j\beta_1)}\right]_{s=-j\beta_1} = \frac{I_m\omega}{-2C_1\beta_1^2} = -\frac{I_m\omega L_1}{2},$$

$$K_3 \triangleq \left[\frac{I_m\omega}{C_2(s^2+\beta_2^2)}\right]_{s=0} = I_m\omega L_2 = 19.2 \times 10^3,$$

$$K_4 \triangleq \left[\frac{I_m\omega}{C_2 s(s+j\beta_2)}\right]_{s=-j\beta_2} = -\frac{I_m\omega L_2}{2}.$$

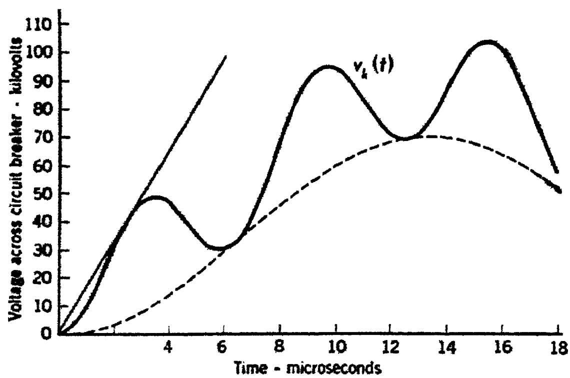


FIG. 7·23. Recovery voltage after interruption of short-circuit current.

Upon carrying out the operations indicated in equation 80, the recovery voltage for $0 \leq t \ll 2\pi/\omega$, with t in microseconds in the second line, is

$$\begin{aligned} v_K(t) &\triangleq I_m\omega L_1(1-\cos\beta_1 t) + I_m\omega L_2(1-\cos\beta_2 t) \\ &= 34.6(1-\cos 0.235t) + 19.2(1-\cos t)\ \text{kv}. \end{aligned} \qquad [81]$$

The form of this recovery voltage is shown in Fig. 7·23. The slope of the steepest of the lines through the origin which are tangent to the curve is the maximum rate-of-recovery of voltage across the circuit breaker under the conditions assumed.

14. Parallel Inverter

The $\mathfrak{L}$-transformation method works best on a problem having one-point initial or boundary conditions. In such a problem the values of the unknown functions and a sufficient number of their derivatives are specified at the origin of time or space. When the specified boundary

conditions are divided between the origin and later instants in time, or other points in space, the $\mathfrak{L}^{-1}$ transformation does not yield a solution with all constants of integration evaluated. Rather it yields a solution containing certain undetermined constants. These are the required but missing initial values. These undetermined constants, or combinations of them, are evaluated by the classical method of substituting the given boundary conditions and solving a set of algebraic equations.

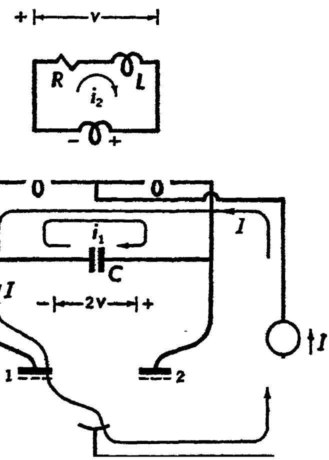


FIG. 7·24. Parallel inverter with inductive load.

$R = 10$ ohms	$C = 100$ microfarads	$t_1 = 1/120$ sec
$L = 0.05$ henry	$I = 30$ amperes	

The analysis of an inverter network [MU 1, OS 1, WA 2] for the waveform of the output current will be taken as an example of a problem having two-point boundary conditions. In Fig. 7·24 is shown a simplified network of a parallel inverter with an inductive load. By means of a grid-controlled mercury-arc tube with two anodes and an associated oscillatory circuit, this inverter changes constant unidirectional current into alternating current. Actually the output in the steady state may be considered to be a repeated transient; its waveform depends on the timing and on the circuit constants. The grid-control circuits are not shown, but their function is to supply alternating voltages of fixed frequency to the grids. The latter serve as switches synchronized to operate 180 degrees out of phase so as to permit their respective anodes to alternate in completing the circuit for the return of the constant current to the source. For simplicity it may be assumed that the 3-winding transformer has the same number of turns in each winding,

and has negligible losses, leakage inductances, and exciting current. In brief, it is an ideal transformer serving as a voltage multiplier.

In Fig. 7·24 the arrow directions for the loop currents are shown for the half-cycle in which anode 1 is conducting. I is the constant unidirectional input current, i_1 is an alternating current confined to the loop formed by the capacitance and the two primary windings of the transformer, i_2 is the alternating output current in the load, and v is the voltage drop across the load.

As a result of the assumptions regarding the transformer, the equations of the network written for the branches are

$$2i_1 + i_2 - I = 0, \tag{82}$$

$$v = L\frac{di_2}{dt} + Ri_2, \tag{83}$$

$$i_1 = 2C\frac{dv}{dt}. \tag{84}$$

If v is eliminated from equations 83 and 84,

$$i_1 = 2LC\frac{d^2i_2}{dt^2} + 2RC\frac{di_2}{dt}, \tag{85}$$

and if this result is substituted in equation 82 there is obtained

$$L\frac{d^2i_2}{dt^2} + R\frac{di_2}{dt} + \frac{1}{4C}i_2 = \frac{I}{4C}. \tag{86}$$

Let the grid-control voltages have a period of $2t_1$ seconds. Then the load current and the condenser voltage drop at the beginning and end of a half-cycle satisfy the following relations:

$$i_2(t_1) = -i_2(0),$$

$$2v(t_1) = -2v(0).$$

Substitution of these conditions in equation 83 shows that

$$i_2'(t_1) = -i_2'(0),$$

which is a boundary condition more convenient to use than that for v.

Letting $I_2(s)$ denote the $\mathfrak{L}$ transform of $i_2(t)$, the $\mathfrak{L}$ transformation of equation 86 yields

$$\left(Ls^2 + Rs + \frac{1}{4C}\right)I_2(s) = \frac{I}{4Cs} + Li_2(0)s + Ri_2(0) + Li_2'(0). \tag{87}$$

The solution of equation 87 for $I_2(s)$ is

$$I_2(s) = \frac{a_2 s^2 + a_1 s + a_0}{s(s^2 + b_1 s + b_0)}, \qquad [88]$$

in which $a_2 \triangleq i_2(0)$

$$a_1 \triangleq (R/L)i_2(0) + i_2'(0)$$

$$a_0 \triangleq I/4LC$$

$$b_1 \triangleq R/L = 200$$

$$b_0 \triangleq 1/4LC = 5 \times 10^4.$$

The roots of the characteristic equation,

$$s^2 + b_1 s + b_0 = 0,$$

are $s_1, s_2 \triangleq -100 \pm j200$.

The $\mathfrak{L}^{-1}$ transformation of equation 88 gives for $0 \leqq t \leqq t_1$,

$$i_2(t) \;(=)\; K_0 + K_1 e^{s_1 t} + \bar{K}_1 e^{s_2 t}, \qquad [89]$$

in which

$$K_0 \triangleq \Big[sI_2(s) \Big]_{s=0} = \frac{a_0}{b_0} = I,$$

and K_1 and $\bar{K}_1$ are conjugate complex constants that depend upon the values of $i_2(0)$ and $i_2'(0)$. The solution will be shortened if K_1 and $\bar{K}_1$ rather than $i_2(0)$ and $i_2'(0)$ are chosen as the two constants to be determined.

The expression for $i_2'(t)$ can be found by differentiating equation 89. It is

$$i_2'(t) = s_1 K_1 e^{s_1 t} + s_2 \bar{K}_1 e^{s_2 t}. \qquad [90]$$

If the boundary conditions are now inserted in equations 89 and 90, and terms collected, there is obtained the set of algebraic equations,

$$\left.\begin{aligned} (1 + e^{s_1 t_1})K_1 + (1 + e^{s_2 t_1})\bar{K}_1 &= -2I, \\ s_1(1 + e^{s_1 t_1})K_1 + s_2(1 + e^{s_2 t_1})\bar{K}_1 &= 0. \end{aligned}\right\} \qquad [91]$$

The elimination of $\bar{K}_1$ from this set of equations and solution for K_1 gives

$$K_1 = \frac{2Is_2}{(s_1 - s_2)(1 + e^{s_1 t_1})} = -31.8 e^{j(39.1^\circ - 90^\circ)} \qquad [92]$$

Combining the conjugate-complex functions in equation 89,

$$i_2(t) \;(=)\; K_0 + \mathfrak{R}[2K_1 e^{s_1 t}].$$

Inserting values, the expression for $i_2(t)$ for the half-period $0 \leqq t \leqq t_1$ is

$$i_2(t) \;(=)\; 30 + \mathcal{R}[-63.6e^{-100t}e^{j(200t+39.1°-90°)}]$$
$$= 30 - 63.6e^{-100t} \sin (200t + 39.1°). \qquad [93]$$

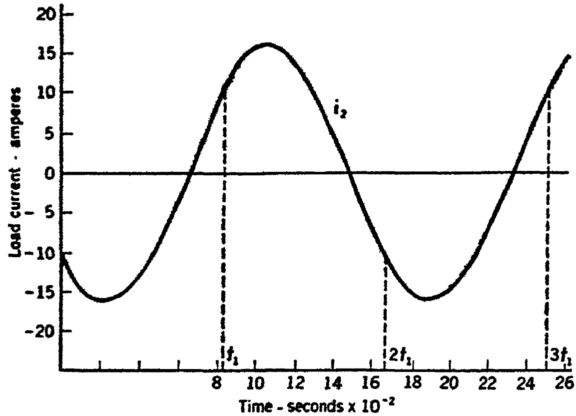


FIG. 7·25. Waveform of alternating current in load of parallel inverter shown in Fig. 7·24

For the half-period $t_1 \leqq t \leqq 2t_1$,

$$i_2(t) = -30 + 63.6e^{-100(t-t_1)} \sin [200(t - t_1) + 39.1°].$$

The waveform of $i_2(t)$ is shown in Fig. 7·25.

PROBLEMS

7·1. Using the $\mathfrak{L}$ transformation, solve the differential equation

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \beta_0^2 x = \frac{d}{dt} (1 - \rho t)e^{-\rho t}.$$

x is to pass through the origin with a positive slope of unity. Assume that $\alpha^2 < \beta_0^2$. Use the abbreviation $\beta_1^2 \triangleq (\rho - \alpha)^2 + (\beta_0^2 - \alpha^2)$.

7·2. Use the $\mathfrak{L}$ transformation to solve the differential equation

$$a_1 \frac{dy}{dt} + a_2 y = a_3 \frac{d}{dt} e^{-\alpha t} \sin \beta t$$

in which a_1, a_2, a_3, α, and β are positive real numbers. The initial value of y is $-b$.

7·3. Find the time function for (a) the voltage across elastance S_1 and (b) the voltage across elastance S_2 if the i-d equation for the circuit is

$$L \frac{di}{dt} + Ri + S_{11} \int i\,dt = Ate^{-(R/2L)t}$$

in which $S_{11} \triangleq S_1 + S_2$ and $R = 2\sqrt{LS_{11}}$. When $t = 0$ the initial current in L is 0, the initial voltage across S_1 is $+\gamma$, and the initial voltage across S_2 is 0.

7·4. If the input voltage $v_1(t)$ of a single section (see diagram) of a symmetrical-T, low-pass wave filter is a unit step voltage, what is the form of the resulting output voltage $v_2(t)$? The initial energy stored in the condensers and inductances may be taken as zero.

$$L = 10 \text{ millihenries}, \quad C = 1 \text{ microfarad}, \quad R^2 = L/C.$$

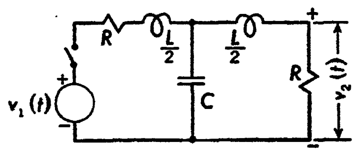


FIG. 7·P4

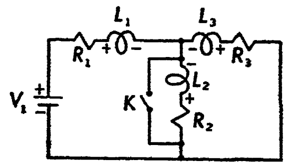


FIG. 7·P5

7·5. Three inductance coils are connected in a 2-loop network as shown in the diagram. There is mutual induction between *each pair* of coils. With the circuit in the steady state, switch K is closed. Find the subsequent current in the battery.

$$\begin{aligned} L_1 &= 50 \text{ millihenries} & M_{23} &= 35 \text{ millihenries} \\ L_2 &= 75 \text{ millihenries} & R_1 &= 5 \text{ ohms} \\ L_3 &= 100 \text{ millihenries} & R_2 &= 15 \text{ ohms} \\ M_{12} &= 30 \text{ millihenries} & R_3 &= 20 \text{ ohms} \\ M_{13} &= 40 \text{ millihenries} & V_1 &= 27.1 \text{ volts} \end{aligned}$$

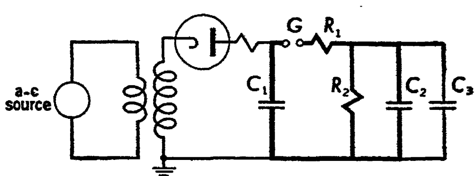


FIG. 7·P6

7·6. The network of a surge generator to produce low surge voltages is illustrated. The half-wave rectifier charging circuit may be neglected in calculations of the discharge of C_1 through the gap G.

(*a*) Taking the initial voltage across C_1 as unity, find the waveform of the voltage applied to C_3.

(*b*) Calculate the time to reach the crest and the time to reach one-half of the crest value on the tail of the surge.

$$\begin{aligned} C_1 &= 1.25 \times 10^{-2} \text{ microfarad} & R_1 &= 160 \text{ ohms} \\ C_2 &= 2.0 \times 10^{-3} \text{ microfarad} & R_2 &= 750 \text{ ohms} \\ C_3 &= 1.0 \times 10^{-4} \text{ microfarad} \end{aligned}$$

7·7. A grid-controlled mercury-vapor discharge tube is used as a synchronous switch in the network shown in diagram *a* to produce recurrent 2.8 × 37-microsecond surges of low voltage for visual oscillograph testing of power apparatus that may be

connected across terminals mn. Here 2.8×37 means 2.8 microseconds to crest and 37 microseconds to half-crest value.

Assume that the grid-controlled tube begins to conduct at each positive maximum of the a-c supply voltage, that the arc drop is constant, and that the voltage drop across R remains constant for 100 microseconds after the tube operates. Assume also that R is sufficiently small to make the charge on C_1 zero prior to each operation of the tube.

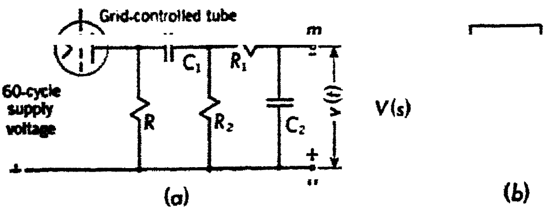


FIG. 7·P7

(a) If $C_2/C_1 = 5.16$, $R_2C_1 = 8.00 \times 10^{-6}$ ohm-farad, and $R_1C_2 = 4.03 \times 10^{-6}$ ohm-farad, find the no-load terminal voltage surge $v(t)$ for one cycle.

(b) Find the impedance function $Z(s)$ to be inserted in the equivalent transform diagram for use in applying Thévenin's theorem. Here $V(s) \triangleq \mathfrak{L}[v(t)]$.

7·8. The equivalent network of one stage of a television amplifier is shown in the diagram. Find its response v_2 to an input voltage v_0 which has the form of a unit step

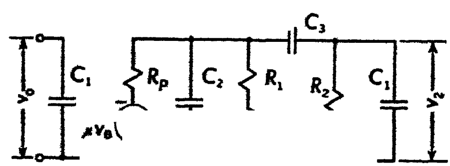


FIG. 7·P8

function. The behavior of the response during the first few microseconds is of major interest.

$R_p = 9 \times 10^5$ ohms	$C_1 = 8 \times 10^{-6}$ microfarad
$R_1 = 10^4$ ohms	$C_2 = 12 \times 10^{-6}$ microfarad
$R_2 = 10^6$ ohms	$C_3 = 0.2$ microfarad
$L = 8 \times 10^{-4}$ henry	$\mu = 2000$

7·9. In a certain rapid-selector system an electronic relay is tripped by the amplified current from a photocell actuated by a light pulse. The triangular pulse of light received by the photocell is shown in diagram a. The equivalent network for the photocell and first stage of the amplifier is shown in diagram b. Find v_3, the input voltage for the second stage. From a practical standpoint, features of particular interest are (1) the time delay introduced by the stage of amplification, and (2) the behavior of v_3 after cessation of the light pulse.

It may be assumed that (1) the pentode operates on the linear part of its characteristic, a change of one volt in the grid voltage causing a change of 10^{-3} ampere in the plate current, (2) the photocell current i_1 is proportional to the light flux received

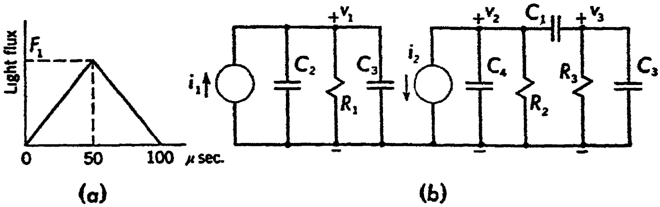


FIG. 7·P9

by the photocell, a flux of F_1 units causing a photocell current of 0.1×10^{-6} ampere, and (3) there is zero time delay in the photocell.

$R_1 = 0.25 \times 10^6$ ohms
$R_2 = 10^5$ ohms
$R_3 = 10^6$ ohms

$C_1 = 0.01$ microfarad
$C_2 = 0.6 \times 10^{-6}$ microfarad
$C_3 = 7 \times 10^{-6}$ microfarad
$C_4 = 12 \times 10^{-6}$ microfarad

7·10. A form of seismometer for use in geophysical exploration is shown. The base, which rests on the earth, has attached to it a circular coil that connects through an amplifier to a recording oscillograph. The coil moves in a radial magnetic field of constant strength supplied by the magnetic core. This core is supported above the base by springs and is restrained so that it can move only vertically. Its motion is damped mechanically. The coil and its load may be assumed to have a combined resistance R and inductance L. The electromechanical coupling constant is U.

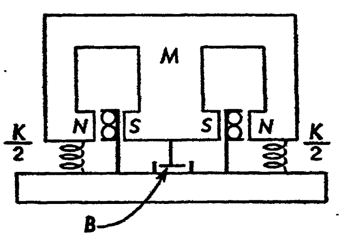


FIG. 7·P10

A sound wave is produced in the earth by the explosion of a charge of dynamite in a hole drilled in the ground. The reflections of this wave from interfaces between layers having different propagation properties and lying as far as 20,000 feet below the surface are of interest. Assume that one of these reflected waves moves the base of the seismometer with a vertical *velocity* $Ae^{-\rho t} \sin \lambda t$. Find the form of the coil current that this wave produces.

In a single system of units the constants are: $M = 0.5$, $K = 4 \times 10^4$, $B = 2.9 \times 10^2$, $R = 370$, $L = 0.35$, and $U = 20$. Let $A = 10^{-5}$; with t in seconds, $\rho = 1$ and $\lambda = 280$.

7·11. The essential elements of an arc-welding generator network are shown in the diagram. The generator is separately excited and differentially compounded. As a good welding machine should be free from serious transients, it is desirable to reduce the magnetic coupling between the exciting circuit and the welding circuit. For this a transformer is used which partially neutralizes the mutual induction within the machine between the main and series fields.

The armature generated voltage $v_g(t)$ due to rotation is

$$v_g(t) = V_0 + k_1(i_1 - I_0) - k_2 i_2,$$

in which V_0 is the normal open-circuit generated voltage and I_0 is the corresponding normal open-circuit main-field current; k_1 and k_2 represent the slope of the field

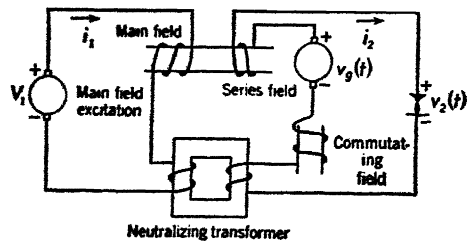


FIG. 7-P11

saturation curve in the working range, k_1 being for the main-field current and k_2 for the armature current. Currents i_1 and i_2 are the actual instantaneous currents in main-field and armature circuits, respectively.

In the main-field circuit the exciting voltage $V_1 = 150$ volts, the normal current $I_0 = 1.3$ amperes, and the total inductance and resistance are, respectively,

$$L_1 = 14 \text{ henries}, \qquad R_1 = 115 \text{ ohms}.$$

In the armature circuit

$$V_0 = 90 \text{ volts} \qquad k_1 = 12 \text{ volts per ampere}$$
$$k_2 = 0.42 \text{ volt per ampere}$$

and the total constants are

$$L_2 = 1.5 \times 10^{-2} \text{ henry}, \qquad R_2 = 2.1 \times 10^{-2} \text{ ohm}.$$

The net mutual inductance is 9.6×10^{-2} henry.

(a) Find the currents in the armature and main field after the arc voltage $v_2(t)$ is suddenly short-circuited. Just previous to the short circuit the arc voltage is 25 volts and the currents are constant.

(b) Find the recovery voltage across the welding electrodes when the short circuit is suddenly opened. It may be assumed that the short-circuit current drops linearly to zero at the rate of 6.30×10^3 amperes per second.

7·12. A radio receiver for ultra-high-frequency reception may have a filter and an amplitude limiter between the aerial circuit and the receiver to reduce interference from static and unwanted stations. For input disturbances below a certain amplitude, however, the output of the filter will be below the response limit of the limiter, the limiter will be inoperative, and hence may be disregarded. For such a condition,

the equivalent circuit of the network, omitting the limiter but including the first filter of the receiver, can be drawn as shown.

Assume that an input disturbance in the aerial circuit produces a unit step voltage across R_a. Calculate the waveform of the resulting *transient* voltage v_2 in the receiver. Consider that R_a and R_b are large, that the tube internal capacitances and plate

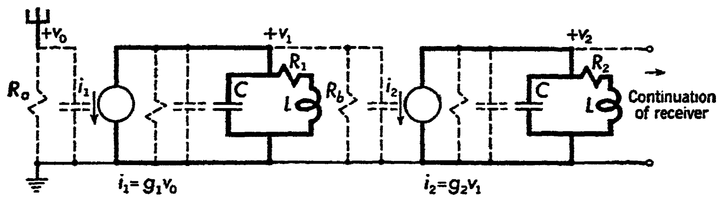


FIG. 7·P12

conductances are small, and that the effect of all of these upon the network response is negligible. The tubes have transconductances g_1 and g_2, respectively. Assume that $R_2 < R_1$ and that the damping constants are very small compared with the characteristic undamped angular frequencies.

7·13. Without actually determining all the roots, answer the following questions for each of the equations given below: (1) Number of roots with positive real parts? (2) Number of roots with negative real parts? (3) Number of roots with zero real parts? What are these roots?

(a) $s^4 + 5s^3 + 13s^2 + 19s + 10 = 0.$
(b) $s^4 + 2s^3 + 4s^2 - 2s - 5 = 0.$
(c) $s^5 + 4s^4 + 7s^3 + 8s^2 + 6s + 4 = 0.$
(d) $s^4 + 2s^3 + s + 2 = 0.$
(e) $s^4 + s^3 - s^2 + s - 2 = 0.$
(f) $s^5 - 9s^3 - 22s^2 - 22s - 8 = 0.$

7·14. In the circuit diagram for a photoelectric potentiometer shown, the unknown voltage is balanced against the drop caused by a known current passing through the

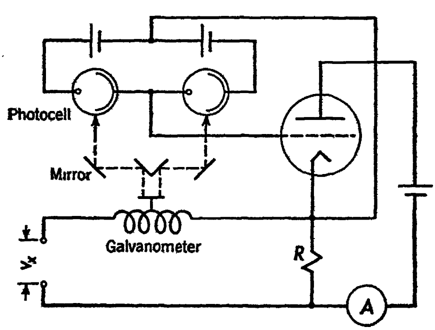


FIG. 7·P14

standard resistor R. Any differential in voltage produces a current which deflects the galvanometer. A beam of light reflected from the galvanometer mirror is divided into two beams by a system of mirrors, each beam falling on a photoelectric cell.

With the galvanometer deflected, the light entering one photocell is increased and that entering the other is decreased. Since this changes the conductivity of the cells there is a change in the grid voltage applied to the vacuum tube which in turn causes a change in the plate current and the necessary correction in the voltage drop across R.

With certain assumptions regarding the photocell characteristics the photocell circuit can be represented by a generator having constant internal resistance and a generated voltage proportional to the galvanometer deflection. For a 1-volt increment in the unknown voltage the necessary correction in plate current can be obtained with a galvanometer current of about 10^{-10} ampere which is entirely negligible in comparison with the plate current. For the vacuum tube the internal plate resistance is R_p and the amplification factor is μ. The galvanometer coil has resistance R_1 and self-inductance L_1. Its moment of inertia is J. The rotational resistance is B and the rotational stiffness is K.

(a) What relations must be satisfied by the system constants if the system is to be stable?

(b) With the system in equilibrium a 1-volt change is made suddenly in the unknown voltage. If the system is stable and slightly oscillatory in the transient state, find the expression for the error in potentiometer balance in terms of literal roots.

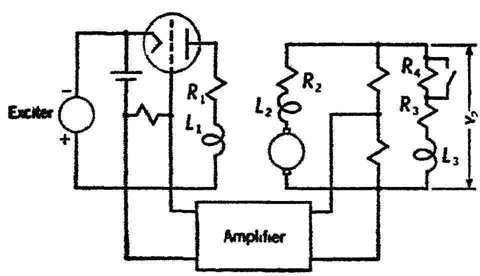


FIG. 7·P15

7·15. A separately excited d-c generator is to have its load voltage v_2 regulated by means of triodes in its field circuit, as shown in the diagram. This is to be accomplished by coupling the load voltage through an amplifier to the grid voltage of the triodes (assumed to be in parallel, but only one of which is shown). If, with the generator operating in the steady state and v_2 equal to the rated voltage, a part R_4 of the load resistance is suddenly short-circuited, find: (a) the critical values of the product, μmk, for which the system just fails to oscillate in the transient state, and (b) the time variation of the *change* in the load voltage when this product has the more favorable of the values found in (a).

It may be assumed that the exciter voltage and the speed of the prime mover remain constant. The triodes may be assumed to be operating on the linear part of their characteristics. The magnetization curve of the generator in the region of operation may be assumed to be straight. The amplifier may be considered as a simple multiplier without distortion or time delay. The total potential divider resistance is very large compared with the other resistances in the armature circuit.

The generator rating is 3.5 kw, 350 volts, 1700 rpm. The dynamic internal plate resistance of the combined triodes is 400 ohms. μ is the amplification factor of the triodes, m is the multiplying factor for the amplifier, and k is the fractional part of

v_2 supplied to the amplifier. A 0.01-ampere change in field current produces a 5-volt change in generated emf.

$$L_1 = 5.2 \text{ henries} \qquad R_2 = 2.3 \text{ ohms}$$
$$L_2 = 0.25 \text{ henry} \qquad R_3 = 41 \text{ ohms}$$
$$L_3 = 0.15 \text{ henry} \qquad R_4 = 29 \text{ ohms}$$
$$R_1 = 500 \text{ ohms}$$

7·16. An automatic controller is used to control the level of the mass M so that the index a attached to M will follow closely the motion of the reference index b (see diagram). The mass is elastically supported by the springs K_1 and K_2 and its motion is damped by viscous friction represented by B. The system is constrained to move only vertically.

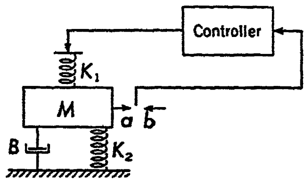


FIG. 7·P16

The spring K_1 connects M with the output of the controller. The controller corrects the level of M by altering the position of the upper end of K_1 an amount equal to C_1 times the deviation of a from b plus C_2 times the time integral of this deviation.

(*a*) In order that the over-all system shall not be unstable and vibrate to destruction when slightly disturbed from balance, what limiting general relations must the four mechanical-system constants and the two proportionality factors of the controller fulfill?

(*b*) Assuming that the relative values of the constants and proportionality factors in any single system of units are as given below, find the resulting deviation of a from b as a function of time if b is suddenly displaced downward a unit amount and held there continuously thereafter. Consider the system previous to this disturbance to have been continuously at a balanced level position.

$$M = 0.13 \qquad K_1 = 4.0 \qquad C_1 = 2.1$$
$$B = 2.8 \qquad K_2 = 22 \qquad C_1 = 32$$

7·17. The diagram shows part of a system for the continuous processing of a rubberized fabric. Individual d-c motors are used to drive rolls 1 and 2, motor 2 being the master motor. When the fabric is taken up by roll 2 more rapidly than it

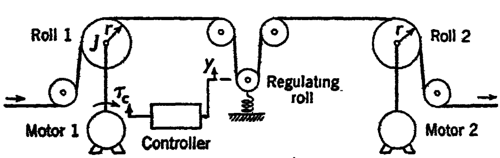


FIG. 7·P17

is delivered by roll 1, the regulating roll will rise, producing a deviation y from the normal position of this roll. A correction of the developed torque of motor 1 is produced by an automatic controller. The equation of control is

$$\tau_c = k_1 y + k_2 \int y\,dt,$$

in which k_1 and k_2 are real constants.

The moment of inertia of motor 1 and roll 1 combined is J. The mass of the regulating roll, the mass of the fabric, and the torque transmitted from roll 2 to roll 1 through the fabric are negligible. The speed-torque characteristic of the load on motor 1 in the region of operation may be considered linear, the increase in torque being B times the increase in speed. Assume that all units are taken from a single system. If the position of the regulating roll is taken as the output variable to be controlled,

(*a*) What is the characteristic equation of the system?

(*b*) What relations should hold among the constants to insure a stable system nonoscillatory in the steady state?

(*c*) If $r = 0.15$, and it is desired to have the poles of the system lie at -10 and $-20 \pm j5$ in the complex plane, what values must the ratios J/B, k_1/J, and k_2/J have?

After the system has been operating without disturbance for a long interval the speed of motor 2 increases in accordance with the function $A(1 - \cos 300t)$ for one cycle and then resumes its former value.

(*d*) Using the constants found in (*c*), what is the resulting vertical motion of the regulating roll?

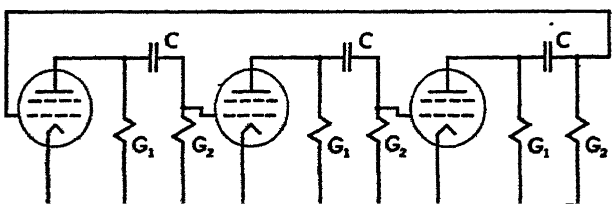


FIG. 7·P18

7·18. A 3-stage RC-coupled vacuum-tube amplifier with feedback can be used to produce an almost sinusoidal oscillation. The slowest ordinary LC oscillations producible with large L's and C's have a period of the order of 1 second, whereas an oscillator with an equivalent network such as shown in the diagram can be made to have a period as great as 30 minutes.

Assume that the system is linear, that C is very much greater than any of the interelectrode capacitances, and that $G_p \ll G_1$ and $G_2 \ll G_1$. Here G_p is the constant plate conductance of each tube. Let μ be the amplification factor of each tube.

(*a*) Give the characteristic equation of the system.

(*b*) Give the relations among the coefficients of this equation if a sustained oscillation is to occur.

(*c*) Give the angular frequency of this oscillation in terms of the constants G_2 and C.

7·19. The network illustrated is in the steady-state condition when switch K closes. Find the resulting current in K in two ways: (1) by the direct method, using the actual initial conditions in L and C, and (2) by use of a substitute source located at the switch. The roots of the characteristic equation are real and different.

7·20. The network illustrated is in the steady-state condition when switch K opens. Find the resulting voltage across K in two ways: (1) by the direct method, which

depends upon a knowledge of the actual condenser voltages at the instant of switch opening, and (2) by use of a substitute source located at the switch.

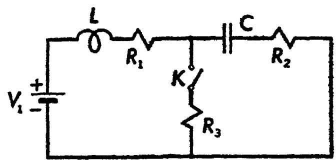


Fig. 7·P19

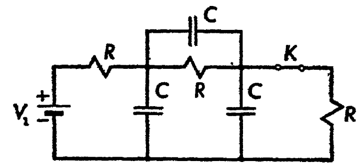


Fig. 7·P20

7·21. A one-line-to-ground short circuit occurs on a 3-phase 60-cycle transmission line 10 miles from the generating station. There is an oil circuit breaker (ocb) between the station transformer bank and the line. The transformers are connected Δ-Y with solidly grounded neutral. The primaries are connected to an "infinite bus." The single-line diagram is shown. Neglect the arc in the ocb and consider that the short-circuit current in the faulted phase is interrupted at a normal zero. Find the system recovery voltage across the opened breaker for the first 100 microseconds after interruption of the current.

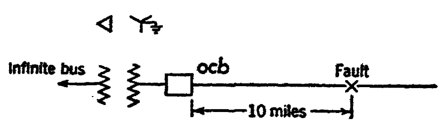


Fig. 7·P21

The line voltage is 140 kv between conductors, and the line reactance is 0.77 ohm per mile. The transformer capacity is 40,000 kva per phase, and the transformer leakage reactance is 8 percent. Neglect the resistances of the transformer, line, and fault. Assume that the reactances to positive-sequence and negative-sequence currents are equal and that the reactance of the line to zero-sequence currents is three times its reactance to positive-sequence currents. The transmission line conductor that was grounded may be represented by its surge impedance of 400 ohms, which enters into the calculations as though it were a resistance. The capacitance to ground of the transformer winding and bushing in the phase that was grounded is 0.01 microfarad. The capacitance of the infinite bus may be taken as infinite.

Note: If the amplitude of the short-circuit current interrupted by the ocb cannot be calculated because of insufficient knowledge of symmetrical components, find the recovery voltage in terms of I_m as a literal constant.

7·22. In the calculation of network recovery voltage the effect of the circuit-breaker arc upon the current being interrupted is usually neglected. If it is to be considered, the following reasoning may be used.

The arc represents a time-varying element in the network. The voltage drop across this element may be considered to be a virtual voltage source that acts upon the fixed part of the network along with the actual sources. The superposition of the currents produced by these sources gives the actual current being interrupted. In general, the arc voltage drop is unknown and an integral equation is needed to express this current, but if an assumption is made about the form of the arc drop the

current can be calculated in the usual way. The important feature is the instant in the voltage cycle at which the arc is extinguished and the current becomes zero. The ensuing build-up of voltage across the circuit breaker can be calculated from the energy storage at this instant and the actual sources in the network.

In the circuit shown in diagram *a* the circuit breaker, in opening, draws an arc across which the voltage drop increases linearly at the rate of 1.2×10^6 volts per second. This arc begins at a normal zero of current, persists until the current is

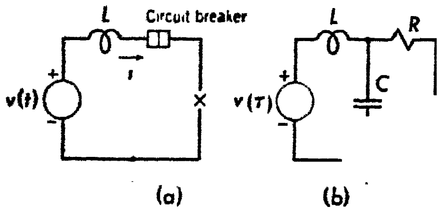


FIG. 7-P22

zero again, and does not restrike. The conducting gases left in the path of the arc provide a leakage path of 10^4 ohms. This is represented by R in diagram *b*. The transformer leakage inductance L is 0.75×10^{-3} henry. The transformer capacitance C to ground is 1.2×10^{-9} farad. Find the circuit recovery voltage, assuming that the voltage across C at the instant of current interruption is the value the arc voltage drop would have at that instant if it continued to increase linearly. For the brief interval of interest, less than 100 microseconds, the source voltage $v(\tau)$ may be considered to remain constant at the value it has at the instant of current interruption. $\tau = 0$ at the instant of current interruption. In part (*a*) of the figure $v(t) = 4000\sqrt{2}\cos 377t$.

Chapter VIII

CERTAIN PROPERTIES OF THE $\mathfrak{L}$ TRANSFORMATION

In Chapters 4 and 5 there have been presented seven theorems stating important properties of the $\mathfrak{L}$ transformation and its inverse. Theorems 1 to 4 were of a general nature covering the basic ideas underlying the $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ transformations. Theorems 5 to 7 stated the linearity property of these transformations and the effects of transforming derivatives and integrals of a function of the real variable. These theorems were essential to the transformation and subsequent solution of linear constant-coefficient i-d equations in one independent variable. For the treatment solely of such equations these theorems are sufficient, but a better understanding of the $\mathfrak{L}$ and $\mathfrak{L}^{-1}$ transformations can be had, and a more effective use made of its concise mathematical expression of complicated physical relations, if certain additional properties of these transformations are known and made part of its working principles [Ka 1, Pa 1, Po 5, 7, Ti 3].

In the present chapter certain additional properties of the $\mathfrak{L}$ transformation will be set forth and for convenience will be stated as theorems. Of these, Theorem 8 is useful in simplifying the inverse transformation of certain transforms by change of argument. Theorem 9 presents a way, different from those so far described, of carrying out the $\mathfrak{L}^{-1}$ transformation of a product. Theorem 10 is concerned with the translation of functions along the axis in the real domain; it will be vital to the treatment later of traveling waves. Theorems 11 and 12 show additional ways of extending a table of transform pairs and of increasing the usefulness of such a table without adding to its length. Theorems 12 and 13 are the bases upon which the transformation method is extended to the solution of equations with two or more independent variables. Theorems 14 and 15 enable one to determine from an $\mathfrak{L}$ transform the behavior of its $\mathfrak{L}^{-1}$ transform at infinity and at the origin without actually carrying out an $\mathfrak{L}^{-1}$ transformation. There is, of course, an overlapping of the fields of usefulness of these various theorems, and the above characterization of them is not to be taken as restrictive.

Supplemental properties of the $\mathfrak{L}$ transformation which are not essential to its application in ordinary cases are presented in Theorems 16 to 20.

1. THEOREM 8, SCALE CHANGE

If the function f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *and* a *is a positive constant, or a second positive variable which is independent of* t *and* s, *then*

$$\mathfrak{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as). \tag{1}$$

In words, this theorem states that division of the variable by a constant, or a second variable, in the real domain goes over into multiplication of both the transform and its variable by this same constant, or second variable, in the complex domain.

The theorem follows from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(\tau)e^{-w\tau}d\tau = F(w),$$

in which w is a complex variable, if τ is first multiplied and divided by a and then t is substituted for $a\tau$, and s for w/a. Here a is a positive constant, or a second positive variable which is independent of τ and w. Carrying out these steps,

$$\int_0^\infty f\left(\frac{a\tau}{a}\right)e^{-(wa\tau/a)}d\left(\frac{a\tau}{a}\right) = F(w), \tag{2}$$

which becomes

$$\int_0^\infty f\left(\frac{t}{a}\right)e^{-st}dt = aF(as), \tag{3}$$

which is the relation

$$\mathfrak{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$$

stated in the theorem.

The new unit of time is $1/a$ times the original unit. If $1 < a$, the f-function is stretched in the t-direction in the ratio $a/1$ and the F-function is modified in two ways, (1) its argument in the s-plane is shrunk radially about the origin in the ratio $1/a$ and (2) its new ordinates are then stretched in the ratio $a/1$. If $0 < a < 1$ the words "stretched" and "shrunk" should be interchanged in the above statement.

Example 1. Starting with the transform pair,

$$\frac{s + 0.50}{(s + 0.50)^2 + \pi^2} \quad \Bigg| \quad e^{-0.50t}\cos \pi t, \qquad 0 \leq t,$$

in which the unit of t is the second, obtain the corresponding pair in which the unit of t will be the half-second.

This can be done by applying Theorem 8, letting $a = 2$. This gives the new pair

$$\frac{s + 0.25}{(s + 0.25)^2 + (\pi/2)^2} \quad \Big| \quad e^{-0.25t} \cos \frac{\pi}{2} t, \qquad 0 \leqq t.$$

Figure 8·1 shows the longitudinal stretching in the ratio of 2/1 of the curve in the real domain which accompanies a radial contraction in the ratio of 1/2 of the geometric pattern formed by the poles in the complex domain.

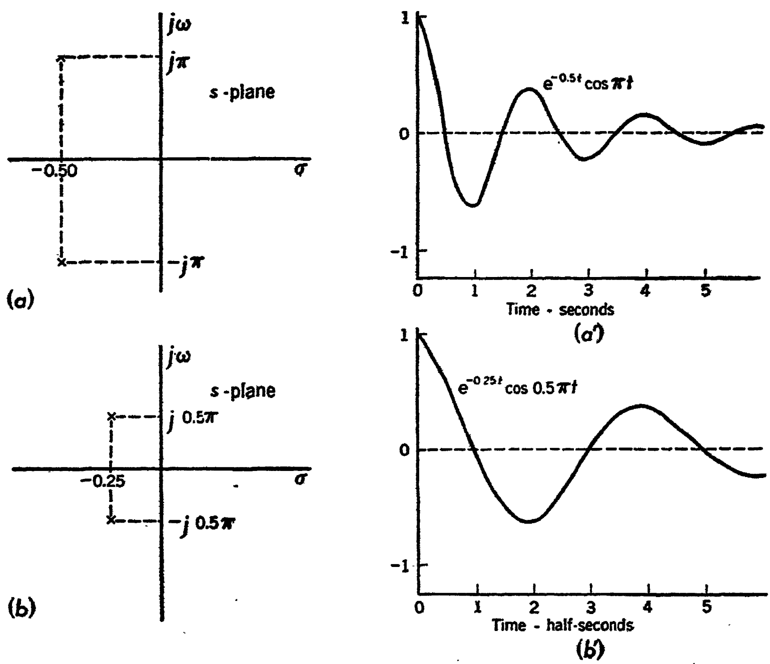


FIG. 8·1. Radial contraction in the complex domain accompanies longitudinal stretching in the real domain.

The following elementary example shows how this theorem can be applied to reduce the labor of carrying out an inverse transformation when the transform contains cumbersome factors. Applications of this principle will be made frequently when partial differential equations are treated.

Example 2. Find $f(t) = \mathfrak{L}^{-1}\left[\dfrac{10^6}{(s + 0.02 \times 10^6)(s + 0.8 \times 10^6)}\right].$

The powers of 10 can be removed by use of Theorem 8. With $a = 10^6$ this gives for $0 \leq t$,

$$f\left(\frac{t}{10^6}\right) = \mathfrak{L}^{-1}\left[\frac{1}{(s + 0.02)(s + 0.8)}\right](=)\,\frac{e^{-0.02t} - e^{-0.8t}}{0.78}. \tag{4}$$

By this step the unit of time for expressing the function in the real domain has been changed from a second to 10^{-6} sec, i.e., to a microsecond. This is shown by setting $t = 1$ in the argument of $f(t/10^6)$.

In terms of the second as the unit of time, the result is

$$f(t) = \frac{e^{-0.02\times 10^6 t} - e^{-0.8\times 10^6 t}}{0.78}, \qquad 0 \leq t.$$

2. Theorem 9, Complex Multiplication

If the functions f_1 *and* f_2 *of* t *are* $\mathfrak{L}$ *transformable and have respectively the* $\mathfrak{L}$ *transforms* F_1 *and* F_2 *of* s, *then*

$$\mathfrak{L}\left[\int_0^t f_1(t - \tau)f_2(\tau)d\tau\right] = F_1(s)F_2(s). \tag{5}$$

The process expressed by the integral will be called *convolution in the real domain*, or *real convolution*, and the functions $f_1(t)$ and $f_2(t)$ will be said to be convolved. This integral operation may be abbreviated to $f_1(t)*f_2(t)$, and read "$f_1(t)$ star $f_2(t)$" [Do 15].

The theorem states that the $\mathfrak{L}$ transformation of the convolution of two functions of the real variable results in the product of the respective transforms of these two functions. Thus convolution in the real domain goes over into multiplication in the complex domain [APPEN C, ME 2].

From the point of view of the direct transformation, the theorem provides another example of the way in which the $\mathfrak{L}$ transformation converts a complicated operation (convolution) in the real domain into a simpler operation (multiplication) in the complex domain. Other examples of this kind were provided by Theorem 6 (real differentiation) and Theorem 7 (real integration).

From the point of view of the inverse transformation, the theorem affords an additional way for carrying out the $\mathfrak{L}^{-1}$ transformation of a function $F(s)$ when that function can be resolved into factors whose inverse transforms can be found readily. Note that for inverse transformation by this method $F(s)$ is first resolved into a *product* of factors. Furthermore, $F(s)$ need not be an algebraic function. This is in contrast to the previous method of inverse transformation presented in Chapter 6 in which $F(s)$ must be algebraic and is first resolved into a *sum* of partial fractions.

As an outline of the proof, let the $\mathfrak{L}$ transforms of the functions

$f_1(t)$ and $f_2(t)$ be $F_1(s)$ and $F_2(s)$, respectively. Then in the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(t)e^{-st}dt = F(s),$$

let

$$f(t) \triangleq \int_0^t f_1(t-\tau)f_2(\tau)d\tau. \qquad [6]$$

The reason for the choice of this special integral for this substitution cannot be apparent at this point. For the present it can be said that the simplicity of the end result will be found to justify the choice. More will be said about this integral later.

The substitution of integral 6 yields the double integral

$$\int_0^\infty \int_0^t f_1(t-\tau)\, f_2(\tau)d\tau \cdot e^{-st}dt = F(s). \qquad [7]$$

Here integration with respect to τ is to be carried out first, then integration with respect to t.

The upper limit of the inner integral can be changed from t to ∞ if the integrand of this integral is multiplied by the step function $u(t-\tau)$, since $f_1(t-\tau)f_2(\tau)\,u(t-\tau)$ is zero for values of τ in the added range, i.e., where τ exceeds t. Equation 7 now becomes

$$\int_0^\infty \int_0^\infty f_1(t-\tau)f_2(\tau)u(t-\tau)d\tau \cdot e^{-st}dt = F(s). \qquad [8]$$

Since the functions $f_1(t)$ and $f_2(t)$ are $\mathfrak{L}$ transformable, the inner integral of equation 8 likewise is $\mathfrak{L}$ transformable. Hence both of the integrals are absolutely convergent and the order of performing the two limit processes represented by these two integrations can be reversed. Changing the order of integration gives

$$\int_0^\infty f_2(\tau) \int_0^\infty f_1(t-\tau)u(t-\tau)e^{-st}dtd\tau = F(s), \qquad [9]$$

in which the integration with respect to t is to be carried out first.

The step function $u(t-\tau)$ makes the value of the new inner integral zero if t is less than τ. The result is the same if the lower limit of this integral is changed to τ, and the function $u(t-\tau)$ omitted. Now let the change of variable $\lambda \triangleq t-\tau$ be made in this inner integral,

$$\int_\tau^\infty f_1(t-\tau)e^{-st}dt = \int_0^\infty f_1(\lambda)e^{-s(\lambda+\tau)}d\lambda = e^{-s\tau}\int_0^\infty f_1(\lambda)e^{-s\lambda}d\lambda. \quad [10]$$

Since it was prescribed that $\mathfrak{L}[f_1(t)] = F_1(s)$, the result found in equation 10 can be written $e^{-s\tau}F_1(s)$, and this on substitution in equation 9 gives

$$F_1(s)\int_0^\infty f_2(\tau)e^{-s\tau}d\tau = F(s). \tag{11}$$

But $\mathfrak{L}[f_2(t)] = F_2(s)$, so equation 11 reduces to

$$F_1(s)F_2(s) = F(s). \tag{12}$$

From this it is seen that equation 7 yields

$$\mathfrak{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] = F_1(s)F_2(s),$$

as stated in the theorem.

It is evident that the arguments of the f_1 and f_2 functions in equation 6 could have been interchanged and the result would have been the same. Consequently

$$\mathfrak{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] = \mathfrak{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = F_1(s)F_2(s). \tag{13}$$

Example 1. Find $\mathfrak{L}^{-1}\left[\dfrac{1}{(s+\alpha)(s+\beta)^2}\right]$ by means of real convolution.

Here the transform $F_1(s)F_2(s)$ is $\dfrac{1}{(s+\alpha)(s+\beta)^2}$. Take as the two factors the functions $\dfrac{1}{s+\alpha}$ and $\dfrac{1}{(s+\beta)^2}$. Since by Table 1, Chapter 4, $\mathfrak{L}^{-1}\left[\dfrac{1}{s+\alpha}\right]$ $(=)\ e^{-\alpha t}$ and $\mathfrak{L}^{-1}\left[\dfrac{1}{(s+\beta)^2}\right]$ $(=)\ te^{-\beta t}$, each for $0 \leqq t$, application of Theorem 9 gives

$$\begin{aligned}
\mathfrak{L}^{-1}\left[\frac{1}{(s+\alpha)(s+\beta)^2}\right] &(=) \int_0^t e^{-\alpha(t-\tau)}\tau e^{-\beta\tau}d\tau \\
&= e^{-\alpha t}\int_0^t \tau e^{(\alpha-\beta)\tau}d\tau \\
&= \frac{e^{-\alpha t}}{(\alpha-\beta)^2}\Big\{e^{(\alpha-\beta)\tau}[(\alpha-\beta)\tau - 1]\Big\}_0^t \\
&= \frac{e^{-\alpha t} + [(\alpha-\beta)t - 1]e^{-\beta t}}{(\alpha-\beta)^2}, \quad 0 \leqq t.
\end{aligned} \tag{14}$$

It may appear from this example that real convolution is a desirable way of carrying out an inverse transformation whenever the inverse transforms of the factors can be recognized. Usually it is a laborious

method and not recommended if the transform is a rational algebraic fraction. For transforms of this type the method of partial fractions is preferable because of its simplicity and the directness with which it yields a solution to transient problems in terms of easily identified steady-state and transient parts. The convolution method is useful, however, as an important part of the general philosophy of transform theory, as will appear in later discussion, and also in the inverse transformation of certain irrational functions by both analytic and machine methods.

Before leaving the general topic of the complex-multiplication theorem two points of special interest will be mentioned.

Theorem 7 (real integration) may be looked upon as a special case of Theorem 9. If $\mathfrak{L}[f(t)] = F(s)$, then by Theorem 9

$$\mathfrak{L}^{-1}\left[\frac{1}{s}F(s)\right](=)\int_0^t u(t-\tau)f(\tau)d\tau = \int_0^t f(\tau)d\tau, \qquad [15]$$

which is Theorem 7. The step function $u(t-\tau)$ could be dropped from the integrand because it is 1 for $\tau < t$ and zero for $t < \tau$.

For the second point it is evident that if the transform is the product of more than two factors the theorem can be applied to the factors grouped in pairs. Thus if $F(s) \triangleq F_1(s)F_2(s)F_3(s)$, and $f_1(t)$, $f_2(t)$, and $f_3(t)$ are the $\mathfrak{L}^{-1}$ transforms, respectively, of $F_1(s)$, $F_2(s)$, and $F_3(s)$, then

$$\mathfrak{L}^{-1}[F(s)]\ (=)\ f_1(t)*f_2(t)*f_3(t), \quad 0 \leqq t. \qquad [16]$$

Furthermore, the order in which the functions are convolved is immaterial.

3. Graphical Interpretation of Real Convolution Integral

A graphical interpretation can be given of the real convolution integral. To have specific functions about which to speak, the integral used in the example of Sec. 2 is chosen for illustration. This integral is

$$\int_0^t e^{-\alpha(t-\tau)}\tau e^{-\beta\tau}d\tau. \qquad [17]$$

The two real functions that are convolved here are $e^{-\alpha t}$ and $te^{-\beta t}$, each for $0 \leqq t$. There can be substituted for these the sectioned functions $e^{-\alpha t}u(t)$ and $te^{-\beta t}u(t)$ since their behavior in the region of interest will be identical with the behavior of the original functions. This substitution is desirable since the sectioned functions are easier to handle graphically.

In Fig. 8·2–*a* is shown the function $e^{-\alpha\tau}u(\tau)$, and in Fig. 8·2–*b* is shown the function $e^{-\alpha(-\tau)}u(-\tau)$. Owing to the presence in the latter of the sectioning function $u(-\tau)$, nonzero values appear only for $\tau \leqq 0$ and the result is a reflection of $e^{-\alpha\tau}u(\tau)$ in the f-axis. In Fig. 8·2–*c* is shown the function $e^{-\alpha(t_1-\tau)}u(t_1 - \tau)$, t_1 being a particular value of t. The presence of the constant t_1 causes a shift of the reflected

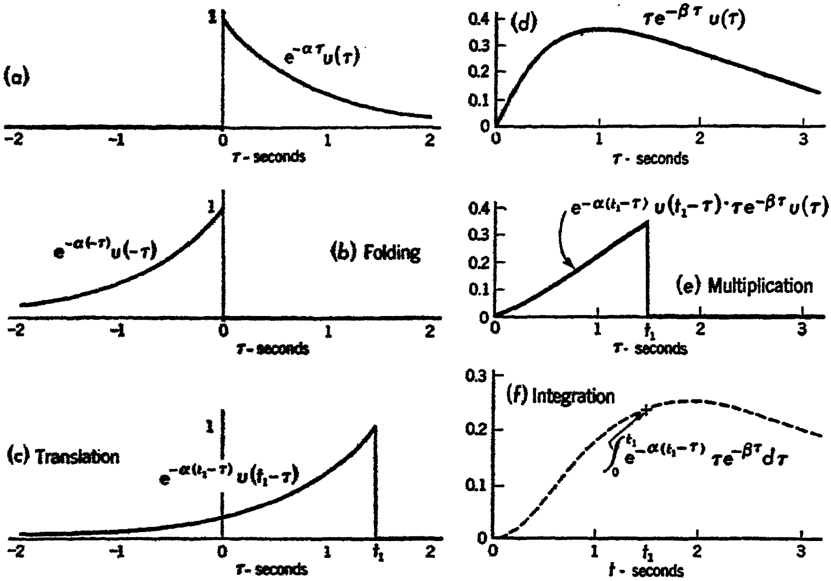


FIG. 8·2. Illustrating the four steps in the graphical evaluation of the convolution integral. $\alpha = 1.1$ and $\beta = 0.1$.

curve to the right by the amount t_1. This shifted curve represents the first factor in integral 17 for the particular instant t_1. To summarize, the replacement of τ by $t_1 - \tau$ first reflects the function in the f-axis and then shifts this reflected function to the right by the amount t_1.

In Fig. 8·2–*d* the second factor $\tau e^{-\beta\tau}u(\tau)$ of integral 17 appears.

The integrand of 17, for $0 \leqq \tau \leqq t$, is the same as the product $e^{-\alpha(t-\tau)}u(t - \tau) \cdot \tau e^{-\beta\tau} \cdot u(\tau)$. This product for the instant t_1 is shown in Fig. 8·2–*e*. The integral 17 with upper limit having the particular value t_1 represents the area under this product curve of Fig. 8·2–*e*. This particular area becomes one point on the curve, Fig. 8·2–*f*, showing the value of integral 17 plotted against t. It is the ordinate at t_1.

It can be seen from this example that "convolution" denotes a mathematical process that can be interpreted graphically by folding,

translating, multiplying, and integrating. It is the English equivalent of the German term " Faltung " [Do 15] and the French term " composition " [Vo 2].

The graphical evaluation of the ordinates of the integral curve (Fig. 8·2–*f*) is a complicated procedure, since for each additional ordinate there must be a new reflection and shifting of the exponential function before a new multiplication of ordinates and integration can be made. In Example 1 of Sec. 2 it was possible to evaluate the convolution integral analytically. Such a method will always be possible when the product transform whose inverse is sought is a rational algebraic function. Analytic treatment is of course possible also in certain cases in which the transform is irrational or even transcendental, but application to such functions will be postponed until Volume 2. Since the graphical treatment can be made where an analytic treatment is extremely difficult or practically impossible, the real convolution integral is an important means of effecting the $\mathfrak{L}^{-1}$ transformation. This will be even more the case as further developments are made of machines that can carry out the convolution process mechanically [Go 2, GR 2, HA 11].

4. FORMULATION OF REAL CONVOLUTION INTEGRAL BY REASONING PHYSICALLY

The real convolution integral can be formulated by reasoning physically, applying the principle of superposition.

In Fig. 8·3–*a* the rectangle A represents a passive linear system whose input and output points are indicated by the two pairs of terminals. Although network terminology appears here, the principles are of general application provided the system is linear. The output response $c(t)$ when the input has the form $u(t)$ will be called the *characteristic time response to a unit step function.*

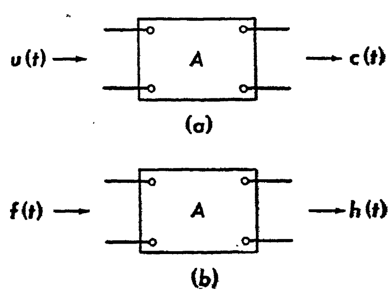


FIG. 8·3

In Fig. 8·3–*b* let $h(t)$ represent the output response of A when the input has the form $f(t)$. It is possible to express $h(t)$ in terms of $f(t)$ and $c(t)$ and remain entirely in the domain of reals. The device for this is the real convolution integral [APPEN C, VA 1].

In Fig. 8·4 an approximation of $f(t)$ is made by adding incremental step functions starting at intervals $\Delta\tau$ along the time axis. When the interval $\Delta\tau$ is nearly zero the increment in f may be approximated by

the product of the slope at some point in the interval and $\Delta\tau$. The response h at any instant t may be considered to be the limit of the sum at that instant of all the individual responses initiated by the preceding incremental step functions as the length of the interval $\Delta\tau \rightarrow 0$. Before passing to the limit, the component due

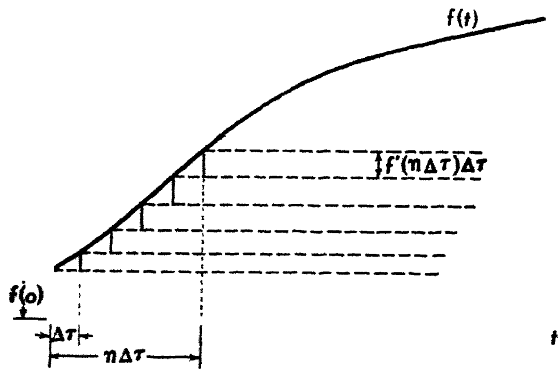


FIG. 8·4. $f(t)$ is approximated by adding step functions.

to the first increment is $f(0)c(t)$, that due to the second increment is $f'(\Delta\tau)\Delta\tau c(t - \Delta\tau)$, that due to the third increment is $f'(2\Delta\tau)\Delta\tau c(t - 2\Delta\tau)$, etc. If there is a limit it is

$$h(t) = f(0)c(t) + \lim_{\Delta\tau \to 0} [f'(\Delta\tau)\Delta\tau c(t - \Delta\tau) + f'(2\Delta\tau)\Delta\tau c(t - 2\Delta\tau) + \cdots + f'(n\Delta\tau)\Delta\tau c(t - n\Delta\tau) + \cdots]. \qquad [18]$$

Let $n\Delta\tau = \tau$. That is, as $\Delta\tau$ decreases let n increase in such a way that their product is finite and equal to τ. Then

$$h(t) = f(0)c(t) + \lim_{\Delta\tau \to 0} \sum_{\tau=0}^{t} f'(\tau)\Delta\tau c(t - \tau)$$

$$= f(0)c(t) + \int_0^t f'(\tau)c(t - \tau)d\tau, \qquad 0 \leq t. \qquad [19]$$

The relation expressed in equation 19 is the basis of the *superposition theorem* [APPEN C.] The real convolution integral is a characteristic feature of this theorem.

If $h(t)$ is known and $f'(t)$ is the unknown, equation 19 becomes an integral equation.

In the domain of reals, equation 19 provides the means of expressing the general relation between a disturbance and a response; it may be

either an ordinary equation or an integral equation. Noteworthy, however, is the simplification that results when the statement of input-output relations is made in the complex domain. To show this let the $\mathfrak{L}$ transforms of $c(t)$, $h(t)$, and $f(t)$ be respectively $C(s)$, $H(s)$, and $F(s)$. Then $\mathfrak{L}[f'(t)] = sF(s) - f(0)$ by Theorem 6. If equation 19 is now transformed, applying Theorem 9, it becomes

$$\begin{aligned} H(s) &= f(0)C(s) + [sF(s) - f(0)]C(s) \\ &= sC(s)F(s). \end{aligned} \qquad [20]$$

Here $sC(s)$ is the system function and it may be designated by the symbol $G(s)$ used previously in Secs. 3 and 4, Chapter 7.

A simple replacement problem will be used as an example. It leads to an integral equation whose solution is much simplified by use of an $\mathfrak{L}$ transformation.

Example 1. The life expectancy within a group of units of a certain fragile device is determined by the following test. A units of this device are put in service on the first of the month, and the number remaining in service thereafter is found to decrease in accordance with the exponential function Ae^{-at}, with t in days and continuous variation assumed.

If it is desired to put B_1 units in service on January 1, and have the total number in service increase daily thereafter in accordance with the growth curve $B - (B - B_1)e^{-bt}$ until ultimately there are B units in service daily, at what rate per day should units be put in service after January 1?

Here the characteristic time response to unit step change is e^{-at}. It is the fractional portion of the sample group of A units still in operation t days after that group was put in service. The output response is the number of units desired in service, $B - (B - B_1)e^{-bt}$. The initial value of the input is B_1. The unknown is $f'(t)$, the rate of supply. Based on equation 19, the relation among these functions is the integral equation

$$B - (B - B_1)e^{-bt} = B_1e^{-at} + \int_0^t f'(\tau)e^{-a(t-\tau)}d\tau, \qquad 0 \leqq t. \qquad [21]$$

Assume that $f'(t)$ is $\mathfrak{L}$ transformable. The $\mathfrak{L}$ transformation of this integral equation gives

$$\frac{B}{s} - \frac{B - B_1}{s + b} = \frac{B_1}{s + a} + \mathfrak{L}[f'(t)]\frac{1}{s + a}. \qquad [22]$$

Solving equation 22 for $\mathfrak{L}[f'(t)]$,

$$\mathfrak{L}[f'(t)] = \frac{[Bb + B_1(a - b)]s + Bab}{s(s + b)}. \qquad [23]$$

The $\mathfrak{L}^{-1}$ transformation of equation 23 yields

$$f'(t) \ (=) \ K_0 + K_1e^{-bt}, \qquad 0 \leqq t,$$

in which

$$K_0 \triangleq \left\{\frac{[Bb + B_1(a - b)]s + Bab}{s + b}\right\}_{s=0} = Ba,$$

$$K_1 \triangleq \left\{\frac{[Bb + B_1(a - b)]s + Bab}{s}\right\}_{s=-b} = (B - B_1)(b - a).$$

Thus the necessary rate of supply of new units per day is

$$f'(t) = Ba + (B - B_1)(b - a)e^{-bt}, \qquad 0 \leqq t. \qquad [24]$$

5. Theorem 10, Real Translation

If the function f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *and if* a *is a non-negative real number, then*

$$\left.\begin{aligned} &(a)\quad \mathfrak{L}[f(t - a)] = e^{-as}F(s) \quad \text{if} \quad f(t - a) = 0, \quad 0 < t < a, \\ &(b)\quad \mathfrak{L}[f(t + a)] = e^{as}F(s) \quad \text{if} \quad f(t + a) = 0, \quad -a < t < 0. \end{aligned}\right\} \qquad [25]$$

In general this theorem states that translation in the t-direction in the real domain goes over into multiplication by an exponential in the complex domain.

The theorem follows from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(\tau)e^{-s\tau}d\tau = F(s)$$

by a substitution of $t - a$ for τ, a being a non-negative real number. This yields

$$\int_a^\infty f(t - a)e^{-s(t-a)}dt = F(s). \qquad [26]$$

Multiplying both sides by e^{-as} gives

$$\int_a^\infty f(t - a)e^{-st}dt = e^{-as}F(s). \qquad [27]$$

Now if $f(t - a) = 0$ for $0 < t < a$, the lower limit of the integral in equation 27 can be changed to zero, giving

$$\int_0 f(t - a)e^{-st}dt = e^{-as}F(s),$$

or

$$\mathfrak{L}[f(t - a)] = e^{-as}F(s), \quad \text{if} \quad f(t - a) = 0, \qquad 0 < t < a, \qquad [28]$$

as stated in part (a) of the theorem.

If the original substitution for τ is $t + a$ instead of $t - a$, the result is

$$\int_{-a}^{\infty} f(t + a)e^{-st}dt = e^{as}F(s). \qquad [29]$$

In case $f(t + a) = 0$ for $-a < t < 0$, the lower limit of the integral can be changed to zero, yielding

$$\int_{0}^{\infty} f(t + a)e^{-st}dt = e^{as}F(s),$$

or

$$\mathfrak{L}[f(t + a)] = e^{as}F(s) \quad \text{if} \quad f(t + a) = 0, \qquad -a < t < 0, \qquad [30]$$

as stated in part (*b*) of the theorem.

If the function $f_1(t)$ is $\mathfrak{L}$ transformable, then a function $f(t)$ of the form $f_1(t)u(t)$ satisfies the conditions for part (*a*), and one of the form $f_1(t)u(t - a)$ satisfies the conditions for part (*b*).

From the point of view of an inverse transformation, Theorem 10 shows that if

$$\mathfrak{L}^{-1}[F(s)] \; (=) \; f(t), \qquad 0 \leqq t,$$

then

$$\mathfrak{L}^{-1}[e^{-as}F(s)] \; (=) \begin{cases} 0 & 0 < t < a \\ f(t - a), & a \leqq t; \end{cases} \qquad [31]$$

but

$$\mathfrak{L}^{-1}[e^{as}F(s)] \; (=) \; f(t + a), \qquad 0 \leqq t, \qquad [32]$$

only if $f(t + a) = 0$ for $-a < t < 0$.

Example 1. By use of Theorem 10 find $\mathfrak{L}[(t - a)^2u(t - a)]$.

Here $f(t - a) \triangleq (t - a)^2u(t - a)$ and is zero for $0 < t < a$. Then by equation 28,

$$\mathfrak{L}[(t - a)^2u(t - a)] = e^{-as}\mathfrak{L}[t^2u(t)] = e^{-as}\frac{2}{s^3}, \qquad 0 < \sigma. \qquad [33]$$

This result can be readily verified by the integral definition of the $\mathfrak{L}$ transformation, i.e.,

$$\int_0^{\infty} (t - a)^2u(t - a)e^{-st}dt = \int_a^{\infty} (t - a)^2e^{-st}dt = e^{-as}\frac{2}{s^3}, \qquad 0 < \sigma. \qquad [34]$$

Example 2. By use of Theorem 10 find $\mathfrak{L}[t^2u(t - a)]$.

Here $f(t - a) \triangleq t^2u(t - a)$ and is zero for $0 < t < a$. Then by equation 28,

$$\mathfrak{L}[t^2u(t - a)] = e^{-as}\mathfrak{L}[(t + a)^2u(t)] = e^{-as}\left(\frac{2}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}\right), \qquad 0 < \sigma. \qquad [35]$$

This result can be readily verified by the integral definition of the $\mathfrak{L}$ transformation, i.e.,

$$\int_0^\infty t^2u(t-a)e^{-st}dt = \int_a^\infty t^2e^{-st}dt = e^{-as}\left(\frac{2}{s^3}+\frac{2a}{s^2}+\frac{a^2}{s}\right), \qquad 0<\sigma. \qquad [36]$$

Example 3. Use the principle of real translation to find the $\mathfrak{L}^{-1}$ transforms of the following two functions in which a is a non-negative real number:

$$(a)\ \frac{1}{s^2}e^{-as}, \quad 0<\sigma; \qquad (b)\ \left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}, \quad 0<\sigma.$$

Apply to (a) the principle expressed in equation 31. Here $F(s) \triangleq s^{-2}$, and $\mathfrak{L}^{-1}[s^{-2}]\ (=)\ t,\ 0 \leqq t$. Then

$$\mathfrak{L}^{-1}\left[\frac{1}{s^2}e^{-as}\right] (=)\ (t-a)u(t-a), \qquad 0 \leqq t. \qquad [37]$$

This function is shown in Fig. 8·5–a; it is a linear function t translated to the right by amount a and sectioned at $t = a$.

(a) (b)

FIG. 8·5. In a the function is translated and sectioned; in b it is only sectioned.

In (b), $F(s) \triangleq as^{-1}+s^{-2}$ and $\mathfrak{L}^{-1}[as^{-1}+s^{-2}]\ (=)\ a+t,\ 0 \leqq t,$ so by equation 31,

$$\mathfrak{L}^{-1}\left[\left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}\right] (=)\ [a+(t-a)]u(t-a) = tu(t-a), \qquad 0 \leqq t. \quad [38]$$

This function is shown in Fig. 8·5–b; it is a linear function t sectioned at the point $t = a$, but not translated.

In these examples the effect of combined translation and sectioning (illustrated by Examples 1 and 3–a) is contrasted with the effect of simply sectioning (illustrated by Examples 2 and 3–b).

If $f(t)$ is a section beginning at $t = 0$ of an $\mathfrak{L}$-transformable periodic function of period a seconds, Theorem 10 can be used to show that its $\mathfrak{L}$ transform $F(s)$ is $F_1(s)/(1-e^{-as})$, in which $F_1(s)$ is the $\mathfrak{L}$ transform of $f_1(t)$, the function in the first period $0 \leqq t \leqq a$ [Mc 2].

By successive applications of Theorem 10, the following table can be developed:

Transform for first period is $F_1(s)$
Transform for second period is $F_1(s)e^{-as}$
Transform for third period is $F_1(s)e^{-2as}$
. . . .
Transform for nth period is $F_1(s)e^{-(n-1)as}$
. . . .

Adding these, the transform for the function is

$$F(s) = F_1(s)\,(1 + e^{-as} + e^{-2as} + \cdots + e^{-(n-1)as} + \cdots)$$
$$= \frac{F_1(s)}{1 - e^{-as}}, \qquad 0 < \sigma. \tag{39}$$

This follows by analogy from the series expansion

$$1 + x + x^2 + x^3 + \cdots = (1 - x)^{-1}, \qquad x^2 < 1.$$

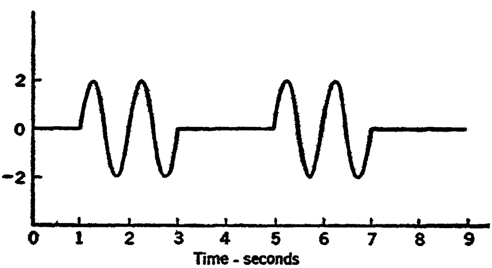


Fig. 8·6. A succession of sinusoidal pulses.

As an example of the application of this principle, the $\mathfrak{L}$ transform of the succession of sinusoidal pulses shown in Fig. 8·6 will be found. Here

$$f(t) \triangleq \begin{cases} 0, & 0 < t < 1 \\ 2 \sin 2\pi t, & 1 < t < 3 \\ 0, & 3 < t < 5 \\ 2 \sin 2\pi t, & 5 < t < 7 \\ \cdots\cdots & \cdots\cdots \end{cases}$$

and the period a of $f(t)$ is 4. For brevity, let $\beta \triangleq 2\pi$. The function in the first period is

$$f_1(t) \triangleq 2 \sin \beta(t - 1)u(t - 1) - 2 \sin \beta(t - 3)u(t - 3), \tag{40}$$

and its transform by Theorem 10 is

$$F_1(s) = \frac{2\beta}{s^2 + \beta^2}(e^{-s} - e^{-3s}). \tag{41}$$

Then by equation 39, the transform of $f(t)$ is

$$F(s) = \frac{2\beta}{s^2 + \beta^2}\cdot\frac{e^{-s} - e^{-3s}}{1 - e^{-4s}} = \frac{2\beta}{s^2 + \beta^2}\cdot\frac{e^{-s}}{1 + e^{-2s}}, \qquad 0 < \sigma. \tag{42}$$

This result can be verified by use of the integral definition of the $\mathfrak{L}$ transformation,

$$\begin{aligned}\int_0^\infty f(t)e^{-st}dt &= 2\int_1^3 \sin\beta t\cdot e^{-st}dt + 2\int_5^7 \sin\beta t\cdot e^{-st}dt + \cdots\\ &\frac{2\beta}{s^2 + \beta^2}\cdot[(e^{-s} - e^{-3s}) + (e^{-5s} - e^{-7s}) +\\ &\frac{2\beta}{s^2 + \beta^2}\cdot\frac{e^{-s}}{1 + e^{-2s}} - \frac{\beta}{s^2 + \beta^2}\cdot\frac{2}{e^{s} + e^{-s}}\\ &- \frac{\beta}{(s^2 + \beta^2)\cosh s}, \qquad 0 < \sigma, \qquad \beta \triangleq 2\pi.\end{aligned} \tag{43}$$

Consider now the converse problem in which

$$\mathfrak{L}^{-1}\left[\frac{\beta}{(s^2 + \beta^2)\cosh s}\right], \quad \text{with} \quad \beta \triangleq 2\pi,$$

is to be found. There are at least two ways in which this inverse transformation can be carried out. The first way depends upon the use of Theorem 10 and repeats in some measure the foregoing relation 43, but it is given here to show the course of the reasoning. The second way will be given in Sec. 6.

Using the expansion in exponentials given in equation 43, the indicated inverse transformation is

$$\mathfrak{L}^{-1}\left[\frac{\beta}{(s^2 + \beta^2)\cosh s}\right] = \mathfrak{L}^{-1}\left[\frac{2\beta}{s^2 + \beta^2}(e^{-s} - e^{-3s} + e^{-5s} - e^{-7s} + \cdots)\right].$$

The infinite series is absolutely convergent for $0 < \sigma$. Proceeding formally, assume that the linearity theorem may be generalized to cover this infinite series and carry out the $\mathfrak{L}^{-1}$ transformation term by term, using Theorem 10 as expressed in equation 31. By this step the order

in which two limit processes are performed is changed. The resulting infinite series is

$$2[\sin \beta(t-1) \cdot u(t-1) - \sin \beta(t-3) \cdot u(t-3) \\ + \sin \beta(t-5) \cdot u(t-5) \\ - \sin \beta(t-7) \cdot u(t-7) + \cdots]. \quad [44]$$

Since $\beta \triangleq 2\pi$, a plot of this series consists of the succession of sinusoidal pulses shown in Fig. 8·6 which was the original function in the time domain. Thus in this case, at least, the inverse transform of an infinite sum of terms is an infinite sum of the inverse transforms of the terms taken separately. A change in the order of the two limit processes of infinite summation and integration such as that made above always requires a check on its validity.

6. Inverse Transformation of a Meromorphic Function

In the previous section a method was given for the $\mathcal{L}^{-1}$ transformation of an $F(s)$ of the type $\beta[(s^2 + \beta^2) \cosh s]^{-1}$ which depended upon expansion in an infinite series of exponential terms in s. A second method will be given here which depends upon an expansion in an infinite sum of partial fractions in s. The same function with $\beta \triangleq 2\pi$ will be used as an example.

First $\cosh s$ will be expanded in an infinite product of linear factors. Since $\cosh s = 0$ for $s = \pm j(\lambda + \frac{1}{2})\pi$ with $\lambda = 0, 1, 2, \cdots$,

$$\cosh s \equiv \left(1 + \frac{s}{j\frac{\pi}{2}}\right)\left(1 - \frac{s}{j\frac{\pi}{2}}\right)\left(1 + \frac{s}{j\frac{3\pi}{2}}\right)\left(1 - \frac{s}{j\frac{3\pi}{2}}\right) \cdots$$

$$\triangleq \prod_{\lambda=0}^{\infty} \left\{\left[1 + \frac{s}{j(\lambda + \frac{1}{2})\pi}\right]\left[1 - \frac{s}{j(\lambda + \frac{1}{2})\pi}\right]\right\}$$

$$\triangleq \prod_{\lambda=0}^{\infty} \left[1 + \frac{s^2}{(\lambda + \frac{1}{2})^2\pi^2}\right] \quad [45]$$

in which the symbol Π indicates the product of factors [KN 2]. Then

$$\frac{\beta}{(s^2 + \beta^2) \cosh s} \equiv \frac{\beta}{(s^2 + \beta^2) \prod_{\lambda=0}^{\infty} \left[1 + \frac{s^2}{(\lambda + \frac{1}{2})^2\pi^2}\right]}. \quad [46]$$

This transform has conjugate first-order poles on the axis of imaginaries at $\pm j\beta$ and at the equally spaced points $\pm j\frac{\pi}{2}$, $\pm j\frac{3\pi}{2}$, etc., there being in all an infinite number of poles.

A function whose only singularities (in the finite part of the plane) are isolated poles is called a *meromorphic function* [Bi 1, Kn 1]. Thus a meromorphic function may have an infinite number of poles. As a consequence it is a generalization of an algebraic rational function. As an example, function 46 is meromorphic.

Since the function 46 is a generalization of a rational function it can be expanded in partial fractions, but here there must be an unlimited number of these fractions in the expansion. This follows from the Mittag-Leffler partial-fraction expansion theorem [Bi 1, Kn 1], which covers the case of an infinite number of poles. The coefficients in this expansion are found in the same way as those for a finite expansion. See Chapter 6, Sec. 5.

The justification of the term-by-term application of the $\mathfrak{L}^{-1}$ transformation to an infinite expansion such as an infinite series or partial-fraction expansion is based on showing that the order of carrying out the two limit processes involved can be validly changed. This usually requires a special study of the particular expansion. It will be sufficient for purposes of solving a physical problem to proceed formally and then justify the result by showing that it satisfies the original equations and boundary conditions.

Returning now to function 46, denoting it by $F(s)$, and carrying out its partial-fraction expansion, there results

$$F(s) = \frac{K_\beta}{s - j\beta} + \frac{\bar{K}_\beta}{s + j\beta} + \sum_{\lambda=0}^{\infty}\left[\frac{K_\lambda}{s - j(\lambda + \frac{1}{2})\pi} + \frac{\bar{K}_\lambda}{s + j(\lambda + \frac{1}{2})\pi}\right], \tag{47}$$

in which

$$K_\beta \triangleq [(s - j\beta)F(s)]_{s=j\beta} \quad \frac{\beta}{2j\beta \cosh (j\beta)} = \frac{1}{2j \cos \beta} = \frac{1}{2j},$$

$$K_\lambda \triangleq \left\{[s - j(\lambda + \tfrac{1}{2})\pi]F(s)\right\}_{s=j(\lambda+\frac{1}{2})\pi} = \frac{\beta}{(s^2 + \beta^2)\dfrac{d}{ds}\cosh s}\Bigg]_{s=j(\lambda+\frac{1}{2})\pi}$$

$$= \frac{\beta}{[\beta^2 - (\lambda + \frac{1}{2})^2\pi^2] \sinh j(\lambda + \frac{1}{2})\pi}.$$

But $\sinh j(\lambda + \frac{1}{2})\pi = j \sin (\lambda + \frac{1}{2})\pi = j(-1)^\lambda$. Hence

$$K_\lambda = \frac{(-1)^\lambda\beta}{j[\beta^2 - (\lambda + \frac{1}{2})^2\pi^2]} = \frac{(-1)^\lambda}{j2\pi\left[1 - \left(\dfrac{2\lambda + 1}{4}\right)^2\right]}.$$

Substituting these coefficients, and combining terms,

$$F(s) = \frac{\beta}{s^2 + \beta^2} + \frac{1}{\pi} \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{1 - \left(\dfrac{2\lambda + 1}{4}\right)^2} \cdot \frac{(\lambda + \frac{1}{2})\pi}{s^2 + (\lambda + \frac{1}{2})^2\pi^2}, \qquad [48]$$

with $\beta \triangleq 2\pi$.

If now it is assumed that the order of inverse transformation and summation can be changed and the $\mathfrak{L}^{-1}$ transformation of both members of equation 48 is carried out, the result is

$$f(t) \ (=) \sin 2\pi t + \frac{1}{\pi} \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{1 - \left(\dfrac{2\lambda + 1}{4}\right)^2} \sin (\lambda + \tfrac{1}{2})\pi t, \quad 0 \leqq t. \qquad [49]$$

Since $f(t)$ can be shown to have function 46 for its $\mathfrak{L}$ transform and to be equal almost everywhere to the section of the original periodic function shown in Fig. 8·6, it is seen that the change in order of limit processes in this example was allowable.

It may be noted in passing that here is a way — different from the usual way — of finding the Fourier series for a periodic function. The steps in the procedure are as follows: (1) The periodic function is sectioned at the origin by multiplying it by $u(t)$. The $\mathfrak{L}$ transform of the first period to the right of the origin is found, and from this is found the $\mathfrak{L}$ transform of the sectioned function, using the relation developed in equation 39. (2) If this transform of the sectioned function is meromorphic it can be expanded in partial fractions. (3) If the order of $\mathfrak{L}^{-1}$ transformation and summation can be changed, the $\mathfrak{L}^{-1}$ transform of the expansion can be found as a sum of trigonometric functions valid for $0 \leqq t$. Since in this non-negative region this sum has the same form as the Fourier series for the original periodic function, it is seen that by dropping the restriction to non-negative values of t the sum becomes the Fourier series for the original periodic function.

For another example of the method, let a voltage $v(t)$ of the form shown in Fig. 8·6 be impressed on a series RL circuit. Assume that the initial current in L is zero. The resulting current will now be found.

Here the applied-voltage transform is

$$V(s) = \frac{\beta}{(s^2 + \beta^2) \cosh s}, \quad \text{with} \quad \beta \triangleq 2\pi.$$

Since the input-admittance function for a series RL circuit is

$$Y(s) = \frac{1}{L(s + \alpha)}, \qquad \alpha \triangleq \frac{R}{L}$$

the resulting current transform is

$$
\begin{aligned}
I(s) &= Y(s)V(s) \\
&= \frac{\beta/L}{(s+\alpha)(s^2+\beta^2)\cosh s}. \qquad [50]
\end{aligned}
$$

The partial-fraction expansion of $I(s)$ is

$$
\begin{aligned}
I(s) = {} & \frac{K_\alpha}{s+\alpha} + \frac{K_\beta}{s-j\beta} + \frac{\bar{K}_\beta}{s+j\beta} \\
& + \sum_{\lambda=0}^{\infty} \left[\frac{K_\lambda}{s - j(\lambda+\frac{1}{2})\pi} + \frac{\bar{K}_\lambda}{s + j(\lambda+\frac{1}{2})\pi} \right], \qquad [51]
\end{aligned}
$$

and

$$
\begin{aligned}
K_\alpha &\triangleq [(s+\alpha)I(s)]_{s=-\alpha} = \frac{\beta/L}{(\alpha^2+\beta^2)\cosh(-\alpha)} \\
&= \frac{}{L(\alpha^2+4\pi^2)\cosh\alpha},
\end{aligned}
$$

$$
\begin{aligned}
K_\beta &\triangleq [(s-j\beta)I(s)]_{s=j\beta} = \frac{\beta/L}{(\alpha+j\beta)2j\beta\cosh j\beta} \\
&= \frac{e^{-j\theta}}{2jL(\alpha^2+4\pi^2)^{\frac{1}{2}}},
\end{aligned}
$$

with $\theta \triangleq \tan^{-1} 2\pi/\alpha$.

$$
\begin{aligned}
K_\lambda &\triangleq \left\{[s - j(\lambda+\tfrac{1}{2})\pi]I(s)\right\}_{s=j(\lambda+\frac{1}{2})\pi} \\
&= \frac{\beta/L}{[\alpha + j(\lambda+\frac{1}{2})\pi][\beta^2 - (\lambda+\frac{1}{2})^2\pi^2]\sinh j(\lambda+\frac{1}{2})\pi} \\
&= \frac{(-1)^\lambda e^{-j\psi_\lambda}}{2j\pi L\left[1 - \left(\frac{2\lambda+1}{4}\right)^2\right][\alpha^2 + (\lambda+\frac{1}{2})^2\pi^2]^{\frac{1}{2}}}
\end{aligned}
$$

with $\psi_\lambda \triangleq \tan^{-1}(\lambda+\frac{1}{2})\pi/\alpha$.

Assuming that the order of summation and inverse transformation can be validly changed, the $\mathfrak{L}^{-1}$ transformation of equation 51 gives

$$
i(t) \;(=)\; K_\alpha e^{-\alpha t} + \mathcal{I}[2jK_\beta e^{j\beta t}] + \sum_{\lambda=0}^{\infty} \mathcal{I}[2jK_\lambda e^{j(\lambda+\frac{1}{2})\pi t}], \qquad 0 \leqq t. \qquad [52]
$$

Substituting for the K's, there results

$$i(t) \ (=) \ \frac{2\pi e^{-\alpha t}}{L(\alpha^2 + 4\pi^2) \cosh \alpha} + \frac{\sin (2\pi t - \theta)}{L(\alpha^2 + 4\pi^2)^{\frac{1}{2}}}$$

$$+ \frac{1}{\pi L} \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda \sin [(\lambda + \frac{1}{2})\pi t - \psi_\lambda]}{\left[1 - \left(\frac{2\lambda + 1}{4}\right)^2\right][\alpha^2 + (\lambda + \frac{1}{2})^2\pi^2]^{\frac{1}{2}}}, \quad 0 \leqq t. \quad [53]$$

The first term is the transient, and the remaining terms are the steady state. For values of $t \gg \alpha^{-1}$ the transient term is negligible, and $i(t)$ is expressed by an infinite series of sinusoidal functions.

That equation 53 is the correct solution for $i(t)$ and that the change in order of limit processes is legitimate in this case can be substantiated by showing that $i(t)$ satisfies the problem's differential equation,

$$L\frac{di}{dt} + Ri = v(t),$$

and satisfies the initial condition that $i(0) = 0$. In verifying equation 53 by substitution in the differential equation, the result given in equation 49 can be used for $v(t)$. In showing that $i(0) = 0$, function 46 and equation 48 with s replaced by α will be found useful.

7. THEOREM 11, COMPLEX TRANSLATION

If the function f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *and if* a *is a complex number with non-negative real part, then*

$$\left.\begin{aligned} &(a) \ \ \mathfrak{L}[e^{-at}f(t)] = F(s + a), \\ &(b) \ \ \mathfrak{L}[e^{at}f(t)] = F(s - a). \end{aligned}\right\} \quad [54]$$

This theorem states that multiplication by an exponential in the real domain goes over into the complex domain as a translation of the function, and in particular its singularities and zeros.

The theorem follows from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(t)e^{-wt}dt = F(w),$$

in which w is a complex variable, by replacing w by $s + a$ to obtain

$$\int_0^\infty f(t)e^{-(s+a)t}dt = \int_0^\infty [f(t)e^{-at}]e^{-st}dt = F(s + a). \quad [55]$$

That is, from equation 55,

$$\mathfrak{L}[f(t)e^{-at}] = F(s + a), \quad [56]$$

as stated in part (a) of the theorem.

Part (b) of the theorem follows immediately if in the original equation w is replaced by $s - a$ instead of $s + a$.

Forms of this theorem useful for inverse transformation are

$$\mathfrak{L}^{-1}[F(s+a)] \ (=)\ e^{-at}\mathfrak{L}^{-1}[F(s)], \qquad 0 \leqq t, \qquad [57\text{–}a]$$

$$\mathfrak{L}^{-1}[F(s-a)] \ (=)\ e^{at}\mathfrak{L}^{-1}[F(s)], \qquad 0 \leqq t, \qquad [57\text{–}b]$$

or their equivalents,

$$\mathfrak{L}^{-1}[F(s)] \ (=)\ e^{at}\mathfrak{L}^{-1}[F(s+a)], \qquad 0 \leqq t, \qquad [58\text{–}a]$$

$$\mathfrak{L}^{-1}[F(s)] \ (=)\ e^{-at}\mathfrak{L}^{-1}[F(s-a)], \qquad 0 \leqq t. \qquad [58\text{–}b]$$

Example 1. Find $\mathfrak{L}[e^{-\alpha t}\cos\beta t]$ from $\mathfrak{L}[\cos\beta t]$ by application of Theorem 11. α is a non-negative real number.

Since $\mathfrak{L}[\cos\beta t] = s/(s^2 + \beta^2), 0 < \sigma$, application of the theorem gives

$$\mathfrak{L}[e^{-\alpha t}\cos\beta t] = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}, \qquad -\alpha < \sigma, \qquad [59]$$

which agrees with pair 7, Table 1, Chapter 4. Note that the poles, the zeros, and the σ_a have all been translated to the left by amount α.

Example 2. Find the $\mathfrak{L}^{-1}$ transform of $s^n/(s+\alpha)^{n+1}$ in which n is a non-negative integer and α is a non-negative real number.

The transform $s^n/(s+\alpha)^{n+1}$ has a pole of order $n+1$ at $-\alpha$. From the discussion in Sec. 1, Chapter 6, it is to be expected that the $\mathfrak{L}^{-1}$ transform will contain the exponential $e^{-\alpha t}$ as a factor. Theorem 11 provides a rule whereby this factor can be written immediately and the transform for the remaining factors formulated directly from the original transform. Thus applying this theorem as expressed in equation 58–b,

$$\mathfrak{L}^{-1}\left[\frac{s^n}{(s+\alpha)^{n+1}}\right] (=)\ e^{-\alpha t}\mathfrak{L}^{-1}\left[\frac{(s-\alpha)^n}{s^{n+1}}\right], \qquad 0 \leqq t. \qquad [60]$$

Expanding $(s-\alpha)^n$ by the binomial theorem and dividing each term by s^{n+1} gives

$$\frac{(s-\alpha)^n}{s^{n+1}} = \frac{1}{s} + \frac{n(-\alpha)}{s^2} + \frac{n(n-1)(-\alpha)^2}{2!s^3} + \cdots + \frac{n!(-\alpha)^k}{(n-k)!k!s^{k+1}} + \cdots$$

$$= \sum_{k=0}^{n} \frac{n!(-\alpha)^k}{(n-k)!k!}\cdot\frac{1}{s^{k+1}}. \qquad [61]$$

The $\mathfrak{L}^{-1}$ transformation of both sides of equation 61 gives, using pair 10, Table 1, Chapter 4,

$$\mathfrak{L}^{-1}\left[\frac{(s-\alpha)^n}{s^{n-1}}\right] (=) \sum_{k=0}^{n} \frac{n!(-\alpha)^k}{(n-k)!k!}\cdot\frac{t^k}{k!}, \qquad 0 \leqq t. \qquad [62]$$

Finally

$$\mathfrak{L}^{-1}\left[\frac{s^n}{(s+\alpha)^{n+1}}\right](=)\, e^{-\alpha t}\sum_{k=0}^{n}\frac{n!(-\alpha)^k}{(n-k)!(k!)^2}t^k, \qquad 0 \leqq t. \qquad [63]$$

The ease with which this process has worked in this example may invite one to use it frequently. Since it depends upon a translation of the singularities in the complex plane, the most favorable case for its application occurs when the singularities of the function to be $\mathfrak{L}^{-1}$ transformed all have the same real coordinate and by one translation can all be moved to the axis of imaginaries. This is not a common case, however, and the method is not in general an advantageous one to use.

For example, suppose this method were applied in the following case in which $\alpha \neq \beta$. Use is made of the result given in equation 61.

$$\mathfrak{L}^{-1}\left[\frac{s^n}{(s+\beta)(s+\alpha)^{n+1}}\right](=)\, e^{-\alpha t}\mathfrak{L}^{-1}\left[\frac{(s-\alpha)^n}{(s+\beta-\alpha)s^{n+1}}\right]$$

$$= e^{-\alpha t}\mathfrak{L}^{-1}\left[\frac{1}{s+\beta-\alpha}\sum_{k=0}^{n}\frac{n!(-\alpha)^k}{(n-k)!k!}\cdot\frac{1}{s^{k+1}}\right], \qquad 0 \leqq t. \qquad [64]$$

The problem now becomes one of finding $\mathfrak{L}^{-1}\left[\frac{1}{(s+\beta-\alpha)s^{k+1}}\right]$, $k = 0, 1, 2, \cdots, n$. It is seen that no appreciable simplification in the problem has resulted from the translation of the poles and the zero to the right by amount α because $e^{-\alpha t}$ is not a factor of *every* term of the function of t. If Theorem 11 is to be used it should be applied only after the original function, which will be denoted by $F(s)$, has been simplified by removal of the pole at $-\beta$. This pole is removed by subtracting from $F(s)$ the term $K_1/(s+\beta)$ in which $K_1 = [(s+\beta)\,F(s)]_{s=-\beta}$.

8. Envelope and Angle Functions of a Response

It is customary to express the response of a system to a driving force as the sum of two components, one a steady-state or a forced component and the other a transient component. In this section a way of expressing the response in terms of an envelope function and an angle function will be given. The method applies when the dominant feature of the response is an oscillation whose period is short compared with the important time constants of the system. It is useful, in particular, when the driving force is a sinusoidal wave whose envelope is varied, i.e., modulated, in a prescribed way. For example, the wave may be modulated by a step function, a rectangular pulse, or a succession of equally spaced rectangular pulses. The response of the system to such a

driving force may be expressed in terms of an envelope function and an angle function. In the development of this method complex translation can be used advantageously, as shown below.

Let the driving force be

$$f(t) = m(t) \sin \omega_0 t, \tag{65}$$

in which $m(t)$ is the "amplitude"-modulating function, and ω_0 is the "carrier" angular frequency. If $f(t)$ and $m(t)$ are $\mathfrak{L}$ transformable and their transforms are $F(s)$ and $M(s)$, respectively, then the $\mathfrak{L}$ transformation of equation 65 gives, by use of Theorem 11,

$$\begin{aligned} F(s) &= \mathfrak{L}[m(t) \sin \omega_0 t] \\ &= \mathfrak{L}\left[\frac{m(t)e^{j\omega_0 t} - m(t)e^{-j\omega_0 t}}{2j}\right] \\ &= \frac{M(s - j\omega_0) - M(s + j\omega_0)}{2j}. \end{aligned} \tag{66}$$

If this driving force is impressed on a linear system whose transfer function is $G(s)$, the response transform is

$$H(s) = G(s)F(s). \tag{67}$$

Substituting for $F(s)$ from equation 66, there is obtained

$$H(s) = \frac{G(s)M(s - j\omega_0) - G(s)M(s + j\omega_0)}{2j}. \tag{68}$$

Representing the inverse transform of $H(s)$ by $h(t)$, the $\mathfrak{L}^{-1}$ transformation of equation 68 gives, with the aid of Theorem 11,

$$\begin{aligned} h(t) \ (=)\ & \mathfrak{L}^{-1}\left[\frac{G(s)M(s - j\omega_0) - G(s)M(s + j\omega_0)}{2j}\right] \\ &= \frac{\mathfrak{L}^{-1}[G(s + j\omega_0)M(s)]e^{j\omega_0 t} - \mathfrak{L}^{-1}[G(s - j\omega_0)M(s)]e^{-j\omega_0 t}}{2j} \\ &= \mathcal{I}\{\mathfrak{L}^{-1}[G(s + j\omega_0)M(s)]e^{j\omega_0 t}\}, \qquad 0 \leq t, \end{aligned} \tag{69}$$

in which the operation of taking the imaginary part is indicated by $\mathcal{I}$.

In general the coefficient of the exponential $e^{j\omega_0 t}$ will be a complex function of a real variable. Let it be indicated by

$$\begin{aligned} \mathfrak{L}^{-1}[G(s + j\omega_0)M(s)] &= p(t) + jq(t) \\ &= a(t)e^{j\phi(t)}, \qquad 0 \leq t, \end{aligned} \tag{70}$$

in which $p(t)$ and $q(t)$ are the real and imaginary parts, respectively, of the "amplitude"; $a(t) \triangleq [p^2(t) + q^2(t)]^{\frac{1}{2}}$, is the envelope function; and $\phi(t) \triangleq \tan^{-1} [q(t)/p(t)]$ is the phase function.

In terms of the functions given in equation 70, the response 69 can be written

$$h(t) \;(=)\; \mathscr{I}[a(t)e^{j\phi(t)}e^{j\omega_0 t}]$$
$$= a(t)\sin[\omega_0 t + \phi(t)], \qquad 0 \leqq t. \qquad [71]$$

Based on expression 71, the response can be represented as the projection on the h-axis of a rotating vector whose magnitude varies as $a(t)$ and whose angle with the reference axis varies as $[\omega_0 t + \phi(t)]$. This vector can be resolved into two component vectors. The first of these is of constant magnitude and rotates uniformly with angular velocity ω_0. The second is of varying magnitude and rotates or swings about the tip of the first vector. This second vector may itself be resolved into several component vectors and may rotate with variable angular velocity, but it will be sufficient here to consider it as one vector. $\phi(t)$ is the phase angle between the resultant vector and a reference axis rotating uniformly with angular velocity ω_0. This rotating reference axis coincides with the stationary reference axis at $t = 0$.

The angular velocity $\omega_r(t)$ of the resultant vector can be found by taking the time derivative of the angle $[\omega_0 t + \phi(t)]$ measured with respect to the stationary axis. That is,

$$\omega_r(t) = \frac{d}{dt}[\omega_0 t + \phi(t)]$$
$$= \omega_0 + \phi'(t) = \omega_0 + \frac{d}{dt}\tan^{-1}\frac{q(t)}{p(t)}$$
$$= \omega_0 + \frac{1}{1 + [q(t)/p(t)]^2}\,\frac{d}{dt}\left[\frac{q(t)}{p(t)}\right]$$
$$= \omega_0 + \frac{p^2(t)}{p^2(t) + q^2(t)}\cdot\frac{p(t)q'(t) - q(t)p'(t)}{p^2(t)}$$
$$= \omega_0 + \frac{p(t)q'(t) - q(t)p'(t)}{a^2(t)}, \qquad 0 \leqq t. \qquad [72]$$

If the modulating function $m(t)$ is a step function, the component vector rotating with angular velocity ω_0 represents the steady-state component of the response, the other vector represents the transient component, and as $t \to \infty$ the angular velocity $\omega_r(t)$ of the resultant vector approaches the angular velocity ω_0 of the steady-state vector. The response has a clearly discernible envelope function $a(t)$ if ω_0 is nearly equal to a characteristic angular frequency of the system and the time constant associated with this characteristic angular frequency is

large compared with the period $2\pi/\omega_0$. The following example will illustrate this case.

In the network of Fig. 8·7 the driving voltage is $v_1(t) = u(t) \sin \omega_0 t$. The resulting voltage $v_2(t)$ across the condenser is to be found, express-

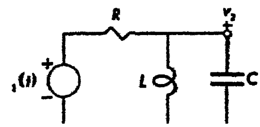


FIG. 8·7. $v_1(t) = u(t) \sin (3.38 \times 10^6 t)$ volts

$R = 10^4$ ohms $\quad C = 3 \times 10^{-3}$ microfarad

$L = 30$ microhenries $\quad \omega_0/\beta_0 = 1.014$

ing it in the form $a(t) \sin [\omega_0 t + \phi(t)]$. The initial energy conditions are those of rest.

Here for simplification the modulating function has been taken as a unit step function, so $M(s) = 1/s$. In accordance with equation 66 the transform of the driving function is

$$V_1(s) \quad \frac{1}{2j}\left(\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right) = \frac{\omega_0}{s^2 + \omega_0^2},$$

which is simply the transform of the sine wave.

For this system the transfer function giving the condenser voltage is obtained by substituting a parallel source and treating the network on the node basis. Let the transform of the condenser voltage $v_2(t)$ be $V_2(s)$. The transfer function is

$$G(s) = \frac{2\alpha s}{(s + \alpha)^2 + \beta^2}, \tag{73}$$

in which

$$\alpha \triangleq 1/2RC = 1.67 \times 10^4, \qquad \beta \triangleq \sqrt{\beta_0^2 - \alpha^2} \approx \beta_0.$$

$$\beta_0 \triangleq 1/\sqrt{LC} = 3.33 \times 10^6,$$

Although the amplitude and phase functions can be found directly from relation 70, it is easier to solve first for the response in the usual way. Thus

$$V_2(s) = G(s)V_1(s)$$

$$= \frac{2\alpha\omega_0 s}{[(s + \alpha)^2 + \beta^2](s^2 + \omega_0^2)} \tag{74}$$

The $\mathcal{L}^{-1}$ transform of equation 74 gives, for $0 \leqq t$,

$$v_2(t)\ (=)\ \mathcal{L}^{-1}\left\{\frac{2\alpha\omega_0 s}{[(s+\alpha)^2+\beta^2](s^2+\omega_0^2)}\right\}$$
$$= \mathcal{I}[Ae^{j(\omega_0 t+\theta)} + Be^{-\alpha t}e^{j(\beta t+\psi)}], \qquad [75]$$

in which

$$A \triangleq \frac{2\alpha\omega_0}{[(\beta_0^2-\omega_0^2)^2+4\alpha^2\omega_0^2]^{\frac{1}{2}}} = 0.328$$

$$B \triangleq \frac{2\alpha\omega_0\beta_0}{\beta[(\beta_0^2-\omega_0^2)^2+4\alpha^2\omega_0^2]^{\frac{1}{2}}} = \frac{\beta_0}{\beta}A \approx 0.328$$

$$\theta \triangleq \frac{\pi}{2} - \tan^{-1}\frac{2\alpha\omega_0}{\beta_0^2-\omega_0^2} = -70.9^\circ$$

$$\psi \triangleq \tan^{-1}\frac{\beta}{-\alpha} - \tan^{-1}\frac{-2\alpha\beta}{\alpha^2-\beta^2+\omega_0^2} = 109.1^\circ.$$

Factoring the exponential $e^{j\omega_0 t}$ from the two terms of equation 75 gives

$$v_2(t)\ (=)\ \mathcal{I}\left[A\left\{e^{j\theta} + \frac{\beta_0}{\beta}e^{-\alpha t}e^{j[(\beta-\omega_0)t+\psi]}\right\}e^{j\omega_0 t}\right]$$
$$= \mathcal{I}\{[p(t)+jq(t)]e^{j\omega_0 t}\}$$
$$= \mathcal{I}\{a(t)e^{j[\omega_0 t+\phi(t)]}\}$$
$$= a(t)\sin[\omega_0 t+\phi(t)], \qquad 0 \leqq t, \qquad [76]$$

in which

$$p(t) \triangleq A\left\{\cos\theta + \frac{\beta_0}{\beta}e^{-\alpha t}\cos[(\beta-\omega_0)t+\psi]\right\},$$

$$q(t) \triangleq A\left\{\sin\theta + \frac{\beta_0}{\beta}e^{-\alpha t}\sin[(\beta-\omega_0)t+\psi]\right\},$$

$$\phi(t) \triangleq \tan^{-1}\frac{q(t)}{p(t)}.$$

The equation of the envelope is

$$a(t) \triangleq [p^2(t)+q^2(t)]^{\frac{1}{2}}$$
$$= A\left\{1+\frac{\beta_0^2}{\beta^2}e^{-2\alpha t} + 2\frac{\beta_0}{\beta}e^{-\alpha t}\cos[(\beta-\omega_0)t+\psi-\theta]\right\}^{\frac{1}{2}}$$
$$= 0.328[1+e^{-0.0333t} - 2e^{-0.0167t}\cos(0.0483t)]^{\frac{1}{2}}, \qquad [77]$$

with t in microseconds. The fractional or relative deviation of the resultant angular frequency from ω_0 is, from relation 72,

$$\frac{\omega_r(t) - \omega_0}{\omega_0} = \frac{1}{\omega_0} \cdot \frac{p(t)q'(t) - q(t)p'(t)}{a^2(t)}$$

$$= 0.0143e^{-0.0167t} \frac{1 - \cos(0.0483t) + 0.345 \sin(0.0483t)}{1 + e^{-0.0333t} - 2e^{-0.0167t} \cos(0.0483t)}, \qquad [78]$$

with t in microseconds. Note that as $t \to \infty$, $a(t) \to A$ and $\omega_r(t) \to \omega_0$, the values respectively of the amplitude and angular frequency of the steady state. Curves of the envelope function $a(t)$ and the fractional deviation of $\omega_r(t)$ from ω_0 are shown in Fig. 8·8.

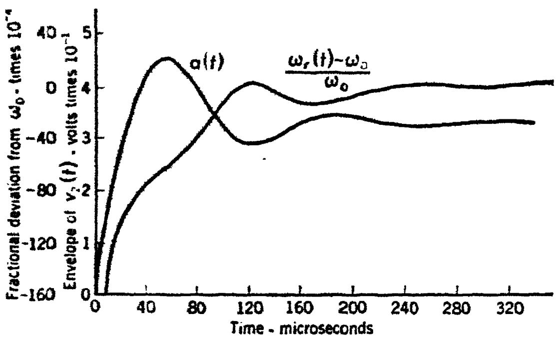


FIG. 8·8. Envelope function of the condenser voltage in the network of Fig. 8·7; also the fractional deviation of its resultant angular velocity from ω_0.

A vector interpretation of the result shown in equation 76 is given in Fig. 8·9. To remove the constant component of the angular velocity, the coordinate axes MM and NN are considered to rotate clockwise with an angular velocity ω_0. The P and Q axes and the steady-state vector $Ae^{j\theta}$ remain stationary. The transient vector rotates clockwise about the tip of the steady-state vector with angular velocity $(\omega_0 - \beta)$. Its tip follows the logarithmic spiral $Be^{-\alpha t}$. The resultant vector has the varying amplitude $a(t)$ and the varying phase angle $\phi(t)$ with respect to the P and Q axes. The instantaneous value of $v_2(t)$ is the projection of the resultant vector on the rotating imaginary axis NN.

9. THEOREM 12, SECOND INDEPENDENT VARIABLE

Let a *be a second variable independent of* t *and* s. *If the function* f(t,a) *is* $\mathfrak{L}$ *transformable with respect to* t *and has the* $\mathfrak{L}$ *transform* F(s,a), *and* $\lim_{a\to a_0}$ f(t,a) *and* $\lim_{a\to a_0}$ F(s,a) *exist, then*

$$\mathfrak{L}_t[\lim_{a\to a_0} f(t,a)] = \lim_{a\to a_0} F(s,a). \qquad [79]$$

This theorem states that the limit process with respect to a is invariant under the transformation from the t domain to the s domain. Stated in other words, the $\mathfrak{L}$ transformation with respect to t and the limit process with respect to a second independent variable are commutative.

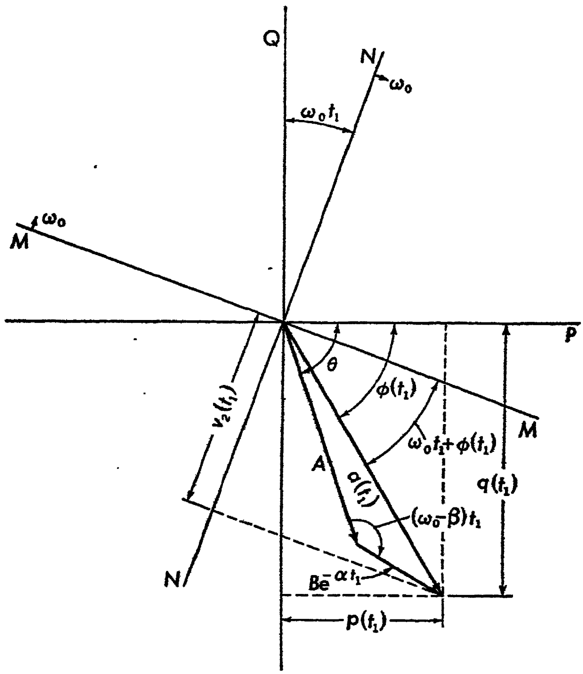


FIG. 8·9. Vector diagram for the instantaneous condenser voltage in the network of Fig. 8·7. Drawn for the instant $t_1 = 50.3$ microseconds.

The theorem follows from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(t,a)e^{-st}dt = F(s,a),$$

and the fact that a is independent of t and s, by taking the limit in both members of the equation as the second independent variable a approaches a_0 since by the hypothesis both limits exist. Thus

$$\lim_{a\to a_0} \int_0^\infty f(t,a)e^{-st}dt = \lim_{a\to a_0} F(s,a), \tag{80}$$

giving

$$\int_0^\infty \lim_{a\to a_0} f(t,a)e^{-st}dt = \lim_{a\to a_0} F(s,a), \tag{81}$$

or

$$\mathfrak{L}_s[\lim_{\alpha \to \alpha_0} f(t,\alpha)] = \lim_{\alpha \to \alpha_0} F(s,\alpha), \tag{82}$$

as stated in the theorem.

By allowing certain independent variables other than the one undergoing transformation to take on limiting values, the property stated in this theorem makes possible the derivation of several particular pairs from more general pairs called key pairs. With this theorem available, tables of transform pairs can be kept brief by listing only a few of these more general pairs.

Example 1. In the following transform pair,

$$\frac{1}{(s+\alpha)(s+\beta)^2} \quad \Bigg| \quad \frac{e^{-\alpha t} + [(\alpha - \beta)t - 1]e^{-\beta t}}{(\alpha - \beta)^2}, \qquad 0 \leqq t, \tag{83}$$

which was derived in Sec. 2, Example 1, α and β can be taken as variables independent of t and s. By use of Theorem 12 find the pairs that result from this pair by letting:

(*a*) $\alpha \to 0$; (*b*) $\beta \to 0$; (*c*) both α and $\beta \to 0$; and (*d*) $\alpha \to \beta$.

(*a*) If $\alpha \to 0$, the pair 83 becomes

$$\frac{1}{s(s+\beta)^2} \quad \Bigg| \quad \frac{1 - (\beta t + 1)e^{-\beta t}}{\beta^2}, \qquad 0 \leqq t. \tag{84}$$

(*b*) If $\beta \to 0$, the pair 83 becomes

$$\frac{1}{s^2(s+\alpha)} \quad \Bigg| \quad \frac{e^{-\alpha t} + \alpha t - 1}{\alpha^2}, \qquad 0 \leqq t. \tag{85}$$

(*c*) If $\alpha \to 0$ in the pair 85, $f(t)$ takes the form 0/0 and must be evaluated by use of l'Hospital's rule. The result is

$$\lim_{\alpha \to 0} \frac{e^{-\alpha t} + \alpha t - 1}{\alpha^2} = \lim_{\alpha \to 0} \frac{t^2 e^{-\alpha t}}{2} = \frac{t^2}{2}, \tag{86}$$

so the derived pair is

$$\frac{1}{s^3} \quad \Bigg| \quad \frac{t^2}{2}, \qquad 0 \leqq t. \tag{87}$$

(*d*) If $\alpha \to \beta$ in the pair 83, $f(t)$ takes the form 0/0 but has the value

$$\lim_{\alpha \to \beta} \frac{e^{-\alpha t} + [(\alpha - \beta)t - 1]e^{-\beta t}}{(\alpha - \beta)^2} = \lim_{\alpha \to \beta} \frac{t^2 e^{-\alpha t}}{2} = \frac{t^2 e^{-\beta t}}{2}. \tag{88}$$

The derived pair is thus

$$\frac{1}{(s+\beta)^3} \quad \Bigg| \quad \frac{t^2 e^{-\beta t}}{2}, \qquad 0 \leqq t. \tag{89}$$

Another application of Theorem 12 can be made in the derivation of the $\mathfrak{L}$ transform of an impulse from the $\mathfrak{L}$ transform of a pulse.

A rectangular pulse of duration a and amplitude a^{-1} is shown in Fig. 8·10. For this pulse $f(t) = \dfrac{u(t) - u(t-a)}{a}$ and its $\mathfrak{L}$ transform is $\dfrac{1 - e^{-as}}{as}$, giving the pair

$$\frac{1 - e^{-as}}{as} \quad \Bigg| \quad \frac{u(t) - u(t-a)}{a}. \tag{90}$$

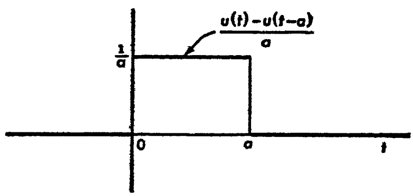


FIG. 8·10. A rectangular pulse which becomes in the limit a unit impulse as $a \longrightarrow 0$.

If $a \longrightarrow 0$ both members of the pair become indeterminate. The

$$\lim_{a \to 0} \frac{1 - e^{-as}}{as} = \lim_{a \to 0} e^{-as} = 1. \tag{91}$$

Let the

$$\lim_{a \to 0} \frac{u(t) - u(t-a)}{a} \triangleq u_1(t), \tag{92}$$

even though strictly speaking this limit does not exist. Of course the limit is also physically unrealizable. In a rigorous sense $u_1(t)$ may be considered to be one of the approximating functions for which a has a very small value. It is common practice to call $u_1(t)$ a singular function and treat it as though it were a unique function. Differentiable [CA 2, 3] rather than rectangular functions are sometimes used for defining $u_1(t)$.

The function $u_1(t)$ will be called a *unit impulse* in view of its similarity to the classical time integral (impulse) of a force in mechanics [TH 3, GI 5, LE 7, SC 3] and the fact that its integral is the unit step function, i.e., $\int_0^t u_1(t)dt = u(t)$. This singular function will be used occasionally in discussions that follow. Wherever it is used, however, it

will be understood that the above reservations apply. Thus there has been derived at least formally a new pair,

$$1 \quad \Big| \quad u_1(t). \qquad [93]$$

When a unit impulse is multiplied by a constant, this coefficient will be considered to be the magnitude of the impulse.

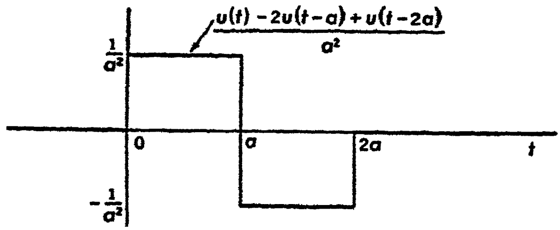


FIG. 8·11. A double rectangular pulse which becomes in the limit a unit doublet impulse as $a \to 0$.

The $\mathfrak{L}$ transform of a doublet impulse can be found from the $\mathfrak{L}$ transform of a double pulse such as shown in Fig. 8·11 by letting $a \to 0$. The pair for this double pulse is

$$\frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} \quad \Big| \quad \frac{u(t) - 2u(t-a) + u(t-2a)}{a^2} \qquad [94]$$

Letting $a \to 0$ leads to indeterminate forms. The

$$\lim_{a \to 0} \frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} = s. \qquad [95]$$

As with the unit impulse let

$$\lim_{a \to 0} \frac{u(t) - 2u(t-a) + u(t-2a)}{a^2} \triangleq u_2(t). \qquad [96]$$

The singular function $u_2(t)$ will be called a *unit doublet impulse*. Its integral is the unit impulse $u_1(t)$, i.e., $\int_0^t u_2(t)dt = u_1(t)$. Thus there has been derived formally a new pair

$$s \quad \Big| \quad u_2(t). \qquad [97]$$

In Sec. 2–*a*, Chapter 6, it was pointed out that any attempt to find the $\mathfrak{L}^{-1}$ transform of an improper rational fraction raises the difficulty of finding the $\mathfrak{L}^{-1}$ transforms of 1, s, s^2, etc. The treatment which had to be postponed there can now be completed, at least formally, by the addition of the impulse terms in accordance with the transform pairs

given in pairs 93 and 97. Thus the $\mathfrak{L}^{-1}$ transform of an improper rational fraction must include singular terms such as the impulse, the doublet impulse, and higher-order impulses [CA 2, 3, GI 5, MC 3].

10. USE OF SPACE IMPULSES IN CALCULATION OF BEAM DEFLECTIONS

In mechanics the study of beam and column deflections includes many cases in which impulse functions can be used to advantage. Although the calculation of deflections caused by static loading does not present a problem in transients, it illustrates in a simple way the usefulness of the concept of space impulses, and certain elementary cases are included here for that reason. When the loading is dynamic, i.e., changing with time, transient vibrations do occur, but the problem then is multidimensional and does not come within the scope of the present discussion.

In the calculation of beam deflections it is customary to express the equation of static equilibrium for a transverse section in terms of the bending moment at this section. For small displacements the approximate equation is a second-order differential equation of the form

$$EI\frac{d^2y}{dx^2} = m(x), \qquad y \triangleq y(x), \tag{98}$$

in which y is the deflection of the beam at x, $m(x)$ is the bending-moment function, E is the modulus of elasticity, and I is the moment of inertia of the transverse cross section. It will be assumed here that the beam has uniform elastic properties and uniform cross section over its length, in which case E and I are constants.

From elementary theory of the strength of materials it is known that the derivative of the bending-moment function is the shear function $v(x)$, and the derivative of the shear function is the force function $f(x)$. Differentiating equation 98 twice with respect to x gives the force equation

$$EI\frac{d^4y}{dx^4} = f(x). \tag{99}$$

Since it is particularly easy to form the force function and the $\mathfrak{L}$ transformation of a fourth-order equation presents no difficulties, equation 99 is a better equation to work with than 98 for the calculation of deflection curves by the method under consideration here.

The origin of the x-coordinate can be taken at any section of the beam, but usually it is most convenient to take it at the left end or, in the case of a cantilever beam, at the fixed end.

The origin of the y-coördinate is taken on a line representing the

neutral axis of the unbent beam. An upward deflection from this reference line at any x is considered positive.

In the formation of the force function, continuous loads are representable by space step functions, whereas concentrated loads and concentrated support reactions are representable by space impulse functions. Forces directed upward are taken as positive. If the beam has a length l, the force function is formed for the region $0 \leqq x \leqq l$, i.e., concentrated loads or support reactions occurring at $x = 0$, or $x = l$ are included in this function. It will be found, however, that those occurring at $x = l$ may as well be omitted from the force function. Beyond their use in the preliminary calculation of the support reactions they do not affect the solution for $0 \leqq x \leqq l$.

The vertical shear $v(x)$ at a section is positive if the resultant of the vertical forces acting on the portion of the beam to the left of the section is upward. The bending moment $m(x)$ at a section is positive if the center of curvature of the deflection curve in that region lies above the curve.

Let the $\mathfrak{L}$ transforms of $y(x)$ and $f(x)$ be $Y(s)$ and $F(s)$, respectively, then the $\mathfrak{L}$ transformation of equation 99 gives

$$EI[s^4Y(s) - y(0)s^3 - y'(0)s^2 - y''(0)s - y'''(0)] = F(s). \quad [100]$$

Solving for $EIY(s)$,

$$EIY(s) = \frac{F(s)}{s^4} + EI\left[\frac{y'''(0)}{s^4} + \frac{y''(0)}{s^3} + \frac{y'(0)}{s^2} + \frac{y(0)}{s}\right]. \quad [101]$$

In this equation

$EIy'''(0) = v(0)$ is the shear at $x = 0$,
$EIy''(0) = m(0)$ is the bending moment at $x = 0$,
$y'(0)$ is the slope at $x = 0$,
$y(0)$ is the deflection at $x = 0$.

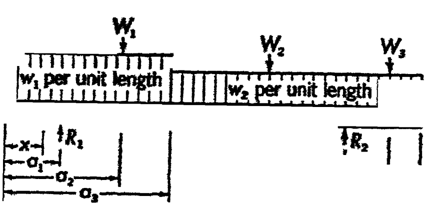


FIG. 8·12. Overhanging beam with distributed and concentrated loads.

This transform equation can be taken as the starting point for the calculation of beam deflections.

Example 1. (*a*) Form the force function for the overhanging beam having the distributed and concentrated loads shown in Fig. 8·12. (*b*) Formulate the boundary conditions.

(a) As a preliminary step, the support reactions R_1 and R_2 are found in the usual way by taking statical moments about the right and left supports.

The origin is taken at the left end of the beam and deflections are measured from the line through the points of support. Then, for $0 \leqq x \leqq a_7$, the force function is

$$f(x) = -w_1[u(x) - u(x - a_3)] - w_2u(x - a_3) + R_1u_1(x - a_1) - W_1u_1(x - a_2) - W_2u_1(x - a_4) + R_2u_1(x - a_5) - W_3u_1(x - a_6). \qquad [102]$$

(b) Four boundary conditions must be found and preferably they should be the shear, bending moment, slope, and deflection at the origin. In this example the shear and bending moment at the origin are easily established but the slope and deflection there cannot be determined before the deflection function $y(x)$ is known. The remaining two boundary conditions are obtained, therefore, from the known deflections at the points of support. Thus the boundary conditions are

$$v(0) = 0, \qquad y(a_1) = 0,$$
$$m(0) = 0, \qquad y(a_5) = 0.$$

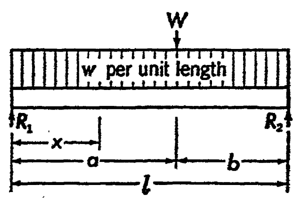


FIG. 8·13.

Taking the origin at the left end of the beam makes the problem a three-point boundary problem.

Example 2. Find the equation of deflection of a uniform beam (Fig. 8·13) supported at the ends and carrying a uniformly distributed load and a concentrated load.

By taking moments about the right support, R_1 is found to be $\dfrac{Wb}{l} + \dfrac{wl}{2}$.

Choosing the origin at the left support, the boundary conditions are

$$v(0) = 0, \qquad y(0) = 0,$$
$$m(0) = 0, \qquad y(l) = 0.$$

The force function is, for $0 \leqq x \leqq l$,

$$f(x) = R_1u_1(x) - wu(x) - Wu_1(x - a). \qquad [103]$$

The $\mathfrak{L}$ transform of $f(x)$ is

$$F(s) = R_1 - \frac{w}{s} - We^{-as}, \qquad [104]$$

in which the third term has been found by application of Theorem 10 to the transform pair 93.

Letting $Y(s)$ be the $\mathfrak{L}$ transform of $y(x)$, the transform equation for the

beam is taken from equation 101 and is

$$EIY(s) = \frac{F(s)}{s^4} + \frac{v(0)}{s^4} + \frac{m(0)}{s^3} + EI\left[\frac{y'(0)}{s^2} + \frac{y(0)}{s}\right]$$

$$= \frac{R_1}{s^4} - \frac{w}{s^5} - \frac{We^{-as}}{s^4} + \frac{EIy'(0)}{s^2}. \qquad [105]$$

Here $EIy'(0)$ is used as an undetermined coefficient to be evaluated later.

The $\mathfrak{L}^{-1}$ transformation of equation 105 gives

$$EIy(x) = \frac{R_1x^3}{3!} - \frac{wx^4}{4!} - \frac{W(x-a)^3}{3!}u(x-a) + EIy'(0)x, \qquad 0 \leqq x \leqq l. \qquad [106]$$

Using the boundary condition at $x = l$ in equation 106, replacing R_1 by its value, and solving for $EIy'(0)$, the result is

$$EIy'(0) = -\frac{wl^3}{4!} + \frac{Wb(b^2 - l^2)}{3!l}. \qquad [107]$$

Substitution from equation 107 in 106 gives the final result

$$EIy(x)\ (=) -\frac{wx^4}{4!} + \left(\frac{Wb}{l} + \frac{wl}{2}\right)\frac{x^3}{3!} + \left[\frac{Wb(b^2 - l^2)}{3!l} - \frac{wl^3}{4!}\right]x$$

$$- \frac{W(x-a)^3}{3!}u(x-a), \qquad 0 \leqq x \leqq l. \qquad [108]$$

The use of space impulse functions has made it possible to express this problem in one differential equation. Two equations would be used in the usual method of analytic solution since the concentrated load divides the beam into two regions having separate differential equations whose solutions must be matched at the boundary between the regions. The more complex the force function the greater is the advantage in using the $\mathfrak{L}$-transformation method.

Example 3. Find the deflection curve of a continuous beam having two unequal spans and bearing a uniformly distributed load, Fig. 8·14.

A relation between R_1 and R_2 is obtained by taking moments about the right support. It is

$$R_1l + R_2b = \frac{wl^2}{2}. \qquad [109]$$

Taking the origin at the left support, the boundary conditions are

$$v(0) = 0, \qquad y(a) = 0,$$

$$m(0) = 0, \qquad y(l) = 0.$$

$$y(0) = 0,$$

The force function for $0 \leqq x \leqq l$ is

$$f(x) = R_1u_1(x) - wu(x) + R_2u_1(x-a), \qquad [110]$$

and its $\mathfrak{L}$ transform is

$$F(s) = R_1 - \frac{w}{s} + R_2 e^{-as}. \tag{111}$$

If $Y(s)$ is the $\mathfrak{L}$ transform of $y(x)$, the transform equation for the beam, by use of equation 101, is

$$EIY(s) = \frac{R_1}{s^4} - \frac{w}{s^5} + \frac{R_2}{s^4} e^{-as} + \frac{EIy'(0)}{s^2}, \tag{112}$$

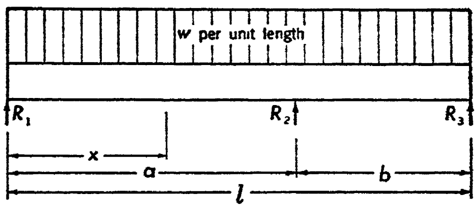


FIG. 8·14.

in which R_1, R_2, and $EIy'(0)$ are undetermined constants. The $\mathfrak{L}^{-1}$ transformation of equation 112 gives

$$EIy(x) \;(=)\; \frac{R_1 x^3}{3!} - \frac{wx^4}{4!} + EIy'(0)x + \frac{R_2(x-a)^3}{3!} u(x-a), \qquad 0 \leqq x \leqq l. \tag{113}$$

Using the boundary conditions at $x = a$ and at $x = l$, the following two algebraic equations are obtained:

$$0 = \frac{R_1 a^2}{3!} - \frac{wa^3}{4!} + EIy'(0), \tag{114}$$

$$0 = \frac{R_1 l^3}{3!} - \frac{wl^4}{4!} + EIy'(0)l + \frac{R_2 b^3}{3!}. \tag{115}$$

Equations 109, 114, and 115 are now solved by algebra for R_1, R_2, and $EIy'(0)$. These are

$$\left.\begin{aligned} R_1 &= \frac{w}{8ab}(l^3 - a^3 - 2lb^2) \\ R_2 &= \frac{wl}{8ab^2}(l^3 + a^3 - 2la^2) \\ EIy'(0) &= \frac{wa}{48b}(-l^3 + a^3 + 2lb^2 + 2a^2b). \end{aligned}\right\} \tag{116}$$

The substitution of expressions 116 in equation 113 gives the equation for $EIy(x)$. Since the interest here is in the method of solution rather than in the beam deflection, the complete equation for $EIy(x)$ will not be written out. A

partial check upon the result may be obtained by assuming that the spans are equal so that $a = b = l/2$. For this case R_1 and R_2 in expressions 116 reduce respectively to $3wl/16$ and $5wl/8$, which can be shown to be correct by other reasoning.

11. Use of Impulses in Interpretation of Real Convolution Integral

For a linear physical system with lumped constants, the general solution for any response transform is an algebraic equation of the form

$$H(s) = G(s)F(s), \qquad [117]$$

in which $H(s)$ is the response transform, $G(s)$ is the system function, and $F(s)$ is the excitation function. Let the $\mathfrak{L}^{-1}$ transforms of $H(s)$, $G(s)$, and $F(s)$ be respectively $h(t)$, $g(t)$, and $f(t)$, $0 \leqq t$. Then using Theorem 9 for the $\mathfrak{L}^{-1}$ transformation of equation 117,

$$h(t) = \int_0^t g(t - \tau)f(\tau)d\tau, \qquad 0 \leqq t. \qquad [118]$$

With the initial conditions of the system all zero and the driving function a unit impulse $u_1(t)$, the response will be represented by $c_1(t)$ and called the *characteristic time response to unit impulse.* Letting $\mathfrak{L}[c_1(t)] = C_1(s)$, equation 117 becomes $C_1(s) = G(s) \cdot 1$ and it is seen that $c_1(t) = g(t)$. Thus the inverse transform of the system function is the characteristic time response to unit impulse.

This result follows also from equation 118 since

$$c_1(t) = \int_0^t g(t - \tau)u_1(\tau)d\tau$$

$$= g(t)\int_0^t u_1(\tau)d\tau = g(t), \qquad 0 \leqq t. \qquad [119]$$

The integrand $g(t - \tau)u_1(\tau)$ is zero except at $\tau = 0$ and there it is $g(t)u_1(\tau)$. The integration is with respect to τ and the integral of the unit impulse is the unit step function.

Taking the point of view that $g(t)$ is the characteristic time response to unit impulse, the convolution integral in equation 118 can be interpreted physically as follows. Let the driving function $f(t)$ be approximated by a succession of elementary rectangular pulses applied at intervals $\Delta\tau$ along the time axis (Fig. 8·15). Each of these pulses when applied to the system produces an elementary response, these responses being superimposed. The approximate total response at time t is the sum at this instant of all the elementary responses started previous to this instant. If the interval $\Delta\tau$ is nearly zero the response

produced by an elementary pulse may be approximated by that produced by an impulse of magnitude equal to the area of the pulse it replaces. Since it was shown in equation 119 that the response to a unit impulse is $g(t)$ the component due to the first impulse is $f(0)\Delta\tau g(t)$, that due to the second impulse is $f(\Delta\tau)\Delta\tau g(t - \Delta\tau)$, and that due to the

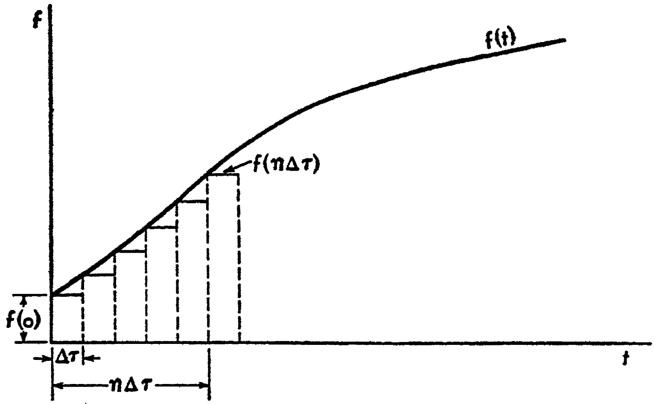


FIG. 8·15. $f(t)$ is approximated by a succession of rectangular pulses.

$(n + 1)$st impulse is $f(n\Delta\tau)\Delta\tau g(t - n\Delta\tau)$. To simplify the notation let $n\Delta\tau = \tau$, that is, as $\Delta\tau$ decreases let n increase in such a way that their product is equal to τ. Finally the response h at any instant t may be considered to be the limit of the sum at t of all the elementary responses initiated by impulses applied between 0 and t as the length of the interval $\Delta\tau \to 0$. Thus

$$h(t) = \lim_{\Delta\tau \to 0} \sum_{\tau=0}^{t} f(\tau)\Delta\tau g(t - \tau)$$

$$= \int_0^t f(\tau)g(t - \tau)d\tau. \qquad [120]$$

Equation 120 is a second form of the superposition theorem (see Sec. 4). It describes the response of the system to an arbitrary driving force in terms of the response of this system to a unit impulse. The form given in Sec. 4 expressed the same result in terms of the response of the system to a unit step function. The form given in equation 120 is somewhat more compact and is easier to use analytically since it is not necessary to differentiate the driving function. On the other hand, if a characteristic time response of the system is to be obtained experimentally it may be simpler to obtain it with a step change than with an impulse. For the latter a pulse must be used whose time duration is small compared with the time constants of the system.

12. Theorem 13, Differentiation with Respect to Second Independent Variable

Let a *be a second variable independent of* t *and* s. *If the function* f(t,a) *is* $\mathfrak{L}$ *transformable with respect to* t *and has the* $\mathfrak{L}$ *transform* F(s,a), *and if* $\frac{\partial}{\partial a} f(t,a)$ *exists, then*

$$\mathfrak{L}_t\left[\frac{\partial}{\partial a} f(t,a)\right] = \frac{\partial}{\partial a} F(s,a). \qquad [121]$$

This theorem states that differentiation with respect to a second independent variable is invariant under transformation from one domain to the other; or stated otherwise, $\mathfrak{L}$ transformation with respect to t and differentiation with respect to a second independent variable are commutative operations.

Since differentiation is a limit process this theorem follows from Theorem 12. In the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(t,a)e^{-st}dt = F(s,a),$$

differentiate both members with respect to the second variable a, i.e.,

$$\frac{\partial}{\partial a}\int_0^\infty f(t,a)e^{-st}dt = \frac{\partial}{\partial a} F(s,a). \qquad [122]$$

Since a is independent of t and s,

$$\int_0^\infty \frac{\partial}{\partial a} f(t,a)e^{-st}dt = \frac{\partial}{\partial a} F(s,a). \qquad [123]$$

That is,

$$\mathfrak{L}_t\left[\frac{\partial}{\partial a} f(t,a)\right] = \frac{\partial}{\partial a} F(s,a), \qquad [124]$$

as stated in the theorem.

The property of the $\mathfrak{L}$ transformation stated in this theorem makes possible the solution by $\mathfrak{L}$ transformation of differential equations with more than one independent variable, and it will be applied frequently in the solution of partial differential equations in Volume 2. Another application, and one which will be illustrated here, is its use in extending a table of function transforms.

Example 1. Find the transform pair that results from differentiation with respect to β of the pair

$$\frac{\beta}{s^2+\beta^2} \quad \Bigg| \quad \sin\beta t, \qquad 0 \leq t.$$

Here

$$\frac{\partial}{\partial \beta}\frac{\beta}{s^2+\beta^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2},$$

and

$$\frac{\partial}{\partial \beta}\sin \beta t = t\cos \beta t.$$

By Theorem 13 these results form the new pair

$$\frac{s^2-\beta^2}{(s^2+\beta^2)^2} \quad \Big| \quad t\cos \beta t, \qquad 0 \leqq t. \qquad [125]$$

Example 2. Find the transform pair that results from differentiation with respect to α of the pair

$$\frac{s+\alpha}{(s+\alpha)^2+\beta^2} \quad \Big| \quad e^{-\alpha t}\cos \beta t, \qquad 0 \leqq t.$$

Here

$$\frac{\partial}{\partial \alpha}\frac{s+\alpha}{(s+\alpha)^2+\beta^2} = -\frac{(s+\alpha)^2-\beta^2}{[(s+\alpha)^2+\beta^2]^2},$$

and

$$\frac{\partial}{\partial \alpha}e^{-\alpha t}\cos \beta t = -te^{-\alpha t}\cos \beta t.$$

By Theorem 13 these results form the new pair

$$\frac{(s+\alpha)^2-\beta^2}{[(s+\alpha)^2+\beta^2]^2} \quad \Big| \quad te^{-\alpha t}\cos \beta t, \qquad 0 \leqq t. \qquad [126]$$

13. THEOREM 14, FINAL VALUE

If the function f(t) *and its first derivative are* $\mathfrak{L}$ *transformable and the* $\mathfrak{L}$ *transform of* f(t) *is* F(s), *and the function* sF(s) *is analytic on the axis of imaginaries and in the right half-plane, then*

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t). \qquad [127]$$

This theorem states that the behavior of $sF(s)$ in the neighborhood of the origin of the complex plane corresponds to the behavior of $f(t)$ as t becomes infinite [PI 10].

This theorem differs from the theorems which have preceded in that it does not express the result of an $\mathfrak{L}$ transformation but states an equality between two particular values — the value of $f(t)$ at ∞ and the value of $sF(s)$ at the origin.

The theorem follows from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty f(t)e^{-st}dt = F(s),$$

by integrating by parts as in the outline of proof of Theorem 6, Sec. 2, Chapter 5, to obtain

$$\int_0^t f'(t)e^{-st}dt = sF(s) - f(0+). \tag{128}$$

Now let s approach 0,

$$\lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0}\,[sF(s) - f(0+)]. \tag{129}$$

Since s is independent of t the order of the limit processes in the left member can be changed giving

$$\int_0^\infty f'(t)dt \quad \lim_{s\to 0}\,[sF(s) - f(0+)], \tag{130}$$

provided each member in equation 130 has a value. The existence of both members of this equation is discussed below. Its left member can be written

$$\lim_{t\to\infty}\int_0^t f'(\tau)d\tau = \lim_{t\to\infty}\,[f(t) - f(0+)]. \tag{131}$$

Substitution of result 131 in equation 130 gives

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{132}$$

From the discussion in Sec. 1, Chapter 6, associating the form of a function of the real variable with the position of the singularities of its transform, it is known that $f'(t)$ will decrease exponentially if all the singularities of its transform $[sF(s) - f(0+)]$ lie to the left of the axis of imaginaries. The condition on $sF(s)$ in the theorem assures this behavior of $f'(t)$. This condition together with the requirement that $f'(t)$ be $\mathfrak{L}$ transformable assures the existence of the limit in equation 131. The conditions on $sF(s)$ also assures the existence of the limit in the right member of equation 130.

The theorem is useful mainly in determining from $F(s)$ the behavior of $f(t)$, as t becomes large, without the necessity of actually carrying out the $\mathfrak{L}^{-1}$ transformation of $F(s)$.

Example 1. To illustrate Theorem 14, use the following transform pairs that have appeared previously:

$$\text{(a)}\quad \frac{}{s(s+\beta)^2} \qquad \frac{1-(1+\beta t)e^{-\beta t}}{\beta^2}, \qquad 0 \leqq t. \tag{84}$$

$$\text{(b)}\quad \frac{1}{s^2(s+\alpha)} \qquad \frac{^{\alpha t} + \alpha t - 1}{\alpha^2}, \qquad 0 \leqq t. \tag{85}$$

(c) $\dfrac{\beta}{s^2+\beta^2}$	$\sin \beta t,$	$0 \leqq t.$	
(d) $\dfrac{(s+\alpha)^2-\beta^2}{[(s+\alpha)^2+\beta^2]^2}$	$te^{-\alpha t}\cos \beta t,$	$0 \leqq t.$	[126]

In these pairs α and β are positive real numbers.

(a) The pole of $sF(s) \triangleq \dfrac{1}{(s+\beta)^2}$ lies in the left half-plane and the theorem is applicable. The $\lim\limits_{s\to 0} \dfrac{1}{(s+\beta)^2} = \dfrac{1}{\beta^2}$, showing that the value approached by $f(t)$ is β^{-2}. This is checked by $\lim\limits_{t\to\infty} \dfrac{1-(1+\beta t)e^{-\beta t}}{\beta^2} = \dfrac{1}{\beta^2}$.

(b) One pole of $sF(s) \triangleq \dfrac{1}{s(s+\alpha)}$ lies on the axis of imaginaries and the theorem does not apply. This is checked by $\lim\limits_{t\to\infty} \dfrac{e^{-\alpha t}+\alpha t-1}{\alpha^2} = \infty$.

(c) Both poles of $sF(s) \triangleq \dfrac{\beta s}{s^2+\beta^2}$ lie on the axis of imaginaries and the theorem does not apply. This is checked by the fact that $\lim\limits_{t\to\infty} \sin \beta t$ does not have a definite value.

(d) Both poles of $sF(s) \triangleq \dfrac{s[(s+\alpha)^2-\beta^2]}{[(s+\alpha)^2+\beta^2]^2}$ lie in the left half-plane and the theorem applies. The $\lim\limits_{s\to 0} \dfrac{s[(s+\alpha)^2-\beta^2]}{[(s+\alpha)^2+\beta^2]^2} = 0$. This is checked by $\lim\limits_{t\to\infty} te^{-\alpha t}\cos\beta t = 0$.

14. Theorem 15, Initial Value

If the function f(t) *and its first derivative are* $\mathfrak{L}$ *transformable and* f(t) *has the* $\mathfrak{L}$ *transform* F(s), *and the* $\lim\limits_{s\to\infty}$ sF(s) *exists, then*

$$\lim_{s\to\infty} sF(s) = \lim_{t \rightharpoonup 0} f(t). \qquad [133]$$

This theorem states that the behavior of $sF(s)$ in the neighborhood of the point at infinity in the complex domain corresponds to the behavior of $f(t)$ in the neighborhood of $0+$ in the real domain [PI 10]. The half arrow with its barb on top signifies that the limit is approached from above, i.e., from the right, passing through positive values of t.

Proof of this theorem proceeds in the same way as that of Theorem 14 to the equation

$$\int_0^\infty f'(t)e^{-st}dt = sF(s) - f(0+). \qquad [128]$$

In equation 128 let s approach ∞,

$$\lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty}[sF(s) - f(0+)]. \qquad [134]$$

The existence of the left member is assured by the hypothesis on the derivative, and when the order of the limit processes is changed the result is zero. The existence of the right member follows from the hypothesis on the limit. Consequently

$$0 = \lim_{s\to\infty}[sF(s) - f(0+)], \qquad [135]$$

or, since $f(0+) \triangleq \lim_{t\to 0} f(t)$, equation 135 can be written

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t),$$

as stated in the theorem.

Like Theorem 14 this theorem expresses an equality which holds between two particular values — the value of $f(t)$ as the origin is approached from the right and the value of $sF(s)$ at the point at infinity. It enables one to establish the initial value of $f(t)$ at $0+$ from $F(s)$ without actually carrying out the $\mathfrak{L}^{-1}$ transformation of $F(s)$. Furthermore, if applied to the transform of the right-hand derivatives of various orders as found by Theorem 6, it gives the initial values of these derivatives at $0+$. Unlike Theorem 14, this property of the $\mathfrak{L}$ transformation holds generally, no restriction on the domain of analyticity of $sF(s)$ being necessary for its application.

Example 1. To illustrate Theorem 15, use the following transform pairs:

(a) $\dfrac{1}{s(s+\beta)^2} \qquad \dfrac{1-(1+\beta t)e^{-\beta t}}{\beta^2} \qquad 0 \leqq t.$ [84]

(b) $\dfrac{s+a_0}{(s+\alpha)^2+\beta^2} \qquad e^{-\alpha t}\left(\cos\beta t + \dfrac{a_0-\alpha}{\beta}\sin\beta t\right), \qquad 0 \leqq t.$ [136]

Pair (b) is obtained by adding and subtracting α in the numerator of $F(s)$ and then using pairs 7 and 6 in Table 1, Chapter 4.

(a) $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\dfrac{1}{(s+\beta)^2} = 0$, showing that the initial value of $f(t)$ is 0. This is checked by $\lim_{t\to 0}\dfrac{1-(1+\beta t)e^{-\beta t}}{\beta^2} = 0.$

(b) $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\dfrac{s(s+a_0)}{(s+\alpha)^2+\beta^2} = 1$, showing that the initial value of $f(t)$ is 1. This is checked by

$$\lim_{t\to 0} e^{-\alpha t}\left(\cos\beta t + \frac{a_0-\alpha}{\beta}\sin\beta t\right) = 1.$$

Example 2. If the $\mathfrak{L}$ transform of $f(t)$ is $(a_1s + a_0)/s(s^2 + b_1s + b_0)$, find the values of $f(t)$ and its first two derivatives at $t = 0+$.

Here

$$\mathfrak{L}[f(t)] = F(s) \triangleq \frac{a_1s + a_0}{s(s^2 + b_1s + b_0)}, \qquad [137]$$

and by Theorem 15,

$$f(0+) = \lim_{s\to\infty} \frac{s(a_1s + a_0)}{s(s^2 + b_1s + b_0)} = 0. \qquad [138]$$

For the first derivative, using Theorem 6,

$$\mathfrak{L}[f'(t)] = sF(s) - f(0+) = \frac{a_1s + a_0}{s^2 + b_1s + b_0} - 0. \qquad [139]$$

Application of Theorem 15 to equation 139 gives

$$f'(0+) = \lim_{s\to\infty} \frac{s(a_1s + a_0)}{s^2 + b_1s + b_0} = a_1. \qquad [140]$$

For the second derivative, by Theorem 6,

$$\begin{aligned}\mathfrak{L}[f''(t)] &= s^2F(s) - f(0+)s - f'(0+)\\ &= \frac{a_1s^2 + a_0s}{s^2 + b_1s + b_0} - 0 - a_1 = \frac{(a_0 - a_1b_1)s - a_1b_0}{s^2 + b_1s + b_0}.\end{aligned} \qquad [141]$$

Application of Theorem 15 to equation 141 gives

$$f''(0+) = \lim_{s\to\infty} \frac{s[(a_0 - a_1b_1)s - a_1b_0]}{s^2 + b_1s + b_0} = a_0 - a_1b_1. \qquad [142]$$

The correctness of these values of $f(t)$, $f'(t)$, and $f''(t)$ at $t = 0+$ can be readily verified. For convenience assume that $b_0 < (b_1/2)^2$; then the poles of $F(s)$ lie at 0 and $-\dfrac{b_1}{2} \pm \left[\left(\dfrac{b_1}{2}\right)^2 - b_0\right]^{\frac{1}{2}} \triangleq s_1, s_2$ and the function is

$$f(t) = \frac{a_0}{b_0} + \frac{a_1s_1 + a_0}{s_1(s_1 - s_2)} e^{s_1t} - \frac{a_1s_2 + a_0}{s_2(s_1 - s_2)} e^{s_2t}, \qquad 0 \leqq t. \qquad [143]$$

Its first two derivatives are

$$f'(t) = \frac{a_1s_1 + a_0}{s_1 - s_2} e^{s_1t} - \frac{a_1s_2 + a_0}{s_1 - s_2} e^{s_2t}, \qquad 0 \leqq t, \qquad [144]$$

$$f''(t) = \frac{s_1(a_1s_1 + a_0)}{s_1 - s_2} e^{s_1t} - \frac{s_2(a_1s_2 + a_0)}{s_1 - s_2} e^{s_2t}, \qquad 0 \leqq t. \qquad [145]$$

At $t = 0+$ equations 143, 144, and 145 reduce respectively to $f(0+) = 0$, $f'(0+) = a_1$, and $f''(0+) = a_0 - a_1b_1$. The initial values found by application of Theorem 15 and given in equations 138, 140, and 142 are in agreement with these results.

15. Multiplication or Division of $F(s)$ by s

The theorems of real differentiation and real integration as introduced in Chapter 5, Secs. 2 and 3, were expressed in a way particularly favorable for use in transforming from the real domain to the complex domain. There are occasions, however, when they must be used in the reverse direction, and it is convenient to have them stated in a different form. Presentation of these alternative forms has been postponed until a knowledge of impulses and the initial-value theorem were available.

THEOREM 6–a. *If* f(t) *and* f′(t) *are* $\mathfrak{L}$ *transformable and* f(0+) = 0, *and* F(s) *is the* $\mathfrak{L}$ *transform of* f(t), *then*

$$sF(s) = \mathfrak{L}[f'(t)]. \qquad [146]$$

Theorem 6–a states that multiplication by s in the complex domain corresponds in the real domain to differentiation. Here, as in Theorem 6, the indicated derivative is a right-hand derivative.

Substantiation of equation 146 depends upon Theorem 6,

$$\mathfrak{L}[f'(t)] = sF(s) - f(0+), \qquad [147]$$

and the condition

$$f(0+) = 0. \qquad [148]$$

It will be recalled from the initial-value theorem that

$$\lim_{s\to\infty} sF(s) = f(0+). \qquad [149]$$

From this it follows that if $sF(s)$ is a rational fraction having a numerator of lower degree than the denominator, $f(0+)$ will be zero; if the numerator and denominator are of equal degree, $f(0+)$ will be finite and different from zero.

Example 1. Starting with the transform pairs

$$(a)\ \frac{1}{(s+\alpha)^2} \qquad te^{-\alpha t}, \qquad 0 \leqq t, \qquad [150]$$

$$(b)\ \frac{s}{s^2+\beta^2} \qquad \cos\beta t, \qquad 0 \leqq t, \qquad [151]$$

find the pairs resulting from multiplication of $F(s)$ by s.

(a) Here $sF(s) \triangleq \dfrac{s}{(s+\alpha)^2}$ which is a proper rational fraction. Correspondingly, $f(0+) = 0$. The right-hand derivative $f'(t) = (1 - \alpha t)e^{-\alpha t}$ and by Theorem 6–a the derived pair is

$$\frac{s}{(s+\alpha)^2} \quad \Bigg| \quad (1-\alpha t)e^{-\alpha t}, \qquad 0 \leqq t. \qquad [152]$$

(b) Here $sF(s) \triangleq \dfrac{s^2}{s^2+\beta^2}$ which is an improper fraction. It reduces on division to $1 - \dfrac{\beta^2}{s^2+\beta^2}$. Correspondingly, $f(0+) \neq 0$, hence Theorem 6–a does not apply. Formally, $\mathfrak{L}^{-1}[1]$ could be interpreted as

$$u_1(t) \triangleq \lim_{a\to 0} \frac{u(t) - u(t-a)}{a}$$

in which case the derived pair is

$$\frac{-\beta^2}{s^2+\beta^2} + 1 \quad \Big| \quad -\beta \sin \beta t + u_1(t), \qquad 0 \leq t.$$

THEOREM 7–a. *If* f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *then*

$$\frac{F(s)}{s} = \mathfrak{L}\left[\int_0^t f(t)dt\right]. \tag{153}$$

The theorem states that division by s in the complex domain corresponds to an integration from 0 to t in the real domain.

Substantiation of equation 153 follows from Theorem 7. That is, since

$$f^{(-1)}(t) = \int f(t)dt = \int_0^t f(t)dt + f^{(-1)}(0+)$$

and

$$\mathfrak{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s},$$

it is seen directly that

$$\frac{F(s)}{s} = \mathfrak{L}\left[\int_0^t f(t)dt\right].$$

Example 1. Starting with the transform pair

$$\frac{s+a_0}{(s+\alpha)^2+\beta^2} \quad \Big| \quad e^{-\alpha t}\left(\cos \beta t + \frac{a_0-\alpha}{\beta}\sin \beta t\right), \qquad 0 \leq t, \tag{136}$$

find the pair resulting from division of $F(s)$ by s.

Here $\dfrac{F(s)}{s} \triangleq \dfrac{s+a_0}{s[(s+\alpha)^2+\beta^2]}$; the definite integral is

$$\int_0^t e^{-\alpha t}\left(\cos \beta t + \frac{a_0-\alpha}{\beta}\sin \beta t\right)dt = \frac{e^{-\alpha t}}{\beta_0^2}\left(\frac{\beta_0^2 - \alpha a_0}{\beta}\sin \beta t - a_0 \cos \beta t\right) + \frac{a_0}{\beta_0^2},$$

in which $\beta_0^2 \triangleq \alpha^2 + \beta^2$. By Theorem 7–*a* the derived pair is

$$\frac{s + a_0}{s[(s+\alpha)^2 + \beta^2]} \quad \Big| \quad \frac{e^{-\alpha t}}{\beta_0^2}\left(\frac{\beta_0^2 - \alpha a_0}{\beta}\sin\beta t - a_0\cos\beta t\right) + \frac{a_0}{\beta_0^2}, \tag{154}$$

$$\beta_0^2 \triangleq \alpha^2 + \beta^2, \qquad 0 \leqq t.$$

16. THEOREM 16, COMPLEX DIFFERENTIATION

If f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *then*

$$\mathfrak{L}[tf(t)] = -\frac{d}{ds}F(s). \tag{155}$$

This theorem states that within a change of sign multiplication by the real variable in the real domain goes over into the complex domain as differentiation with respect to the complex variable. Note that this theorem is analogous to the real differentiation theorem (Theorems 6 and 6–*a*).

The theorem may be proved from the integral definition of the direct transformation

$$\int_0^\infty f(t)e^{-st}dt = F(s)$$

by differentiating both members with respect to s, this being allowable because the hypothesis implies that $F(s)$ is analytic.

$$\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \frac{d}{ds}F(s). \tag{156}$$

Differentiating under the integral sign,

$$\int_0^\infty f(t)\frac{d}{ds}e^{-st}dt = \int_0^\infty -tf(t)e^{-st}dt = \frac{d}{ds}F(s), \tag{157}$$

or $\mathfrak{L}[tf(t)] = -\dfrac{d}{ds}F(s)$ as stated in the theorem.

Example 1. Apply Theorem 16 to the pair

$$\frac{\beta}{s^2 + \beta^2} \qquad \sin\beta t, \qquad 0 \leqq t.$$

Here $\dfrac{d}{ds}\dfrac{\beta}{s^2+\beta^2} = \dfrac{-2\beta s}{(s^2+\beta^2)^2}$, and the derived pair is

$$\frac{2\beta s}{(s^2+\beta^2)^2} \qquad t\sin\beta t, \qquad 0 \leqq t. \tag{158}$$

17. THEOREM 17, COMPLEX INTEGRATION

If f(t) *and* f(t)/t *are* $\mathfrak{L}$*-transformable and the transform of* f(t) *is* F(s), *and if* $\int_s^\infty$ F(s)ds *exists, then*

$$\mathfrak{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds. \qquad [159]$$

Division by the variable in the real domain corresponds to integration with respect to the variable in the complex domain along a path from s to the point at infinity. This theorem is analogous to the real integration theorem (Theorems 7 and 7–a).

To substantiate this theorem, the integral definition of the direct transformation

$$\int_0^\infty f(t)e^{-st}dt = F(s)$$

is integrated with respect to s between the limits s and ∞, i.e.,

$$\int_s^\infty \int_0^\infty f(t)e^{-st}dtds = \int_s^\infty F(s)ds. \qquad [160]$$

A change in the order of carrying out the two limit processes indicated in the left member of equation 160 may be made because $f(t)$ is an $\mathfrak{L}$-transformable function. This gives

$$\int_0^\infty f(t)\int_s^\infty e^{-st}dsdt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \int_s^\infty F(s)ds, \qquad [161]$$

or $\mathfrak{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$ as stated in the theorem.

Example 1. Apply Theorem 17 to the pair

$$\frac{2\beta s}{(s^2+\beta^2)^2} \quad \Bigg| \quad t \sin \beta t, \qquad 0 \leqq t. \qquad [158]$$

Here

$$\int_s^\infty \frac{2\beta s}{(s^2+\beta^2)^2}ds = \frac{-\beta}{s^2+\beta^2}\Bigg|_s^\infty = \frac{\beta}{s^2+\beta^2}, \quad \text{and} \quad \frac{t \sin \beta t}{t} = \sin \beta t.$$

Thus the derived pair is

$$\frac{\beta}{s^2+\beta^2} \quad \Bigg| \quad \sin \beta t, \qquad 0 \leqq t,$$

which is known to be correct.

Example 2. Use Theorem 17 to find the $\mathfrak{L}$ transform of $(e^{-at} - e^{-bt})/t$ with a and b non-negative real numbers and $a < b$.

This may be derived by starting with the pair

$$\frac{1}{s+a} - \frac{1}{s+b} \quad \Bigg| \quad e^{-at} - e^{-bt} \qquad 0 \leqq t,$$

since application of the theorem gives the specified time function $(e^{-at} - e^{-bt})/t$, and gives for its transform

$$\int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = \Big[\ln (s+a) - \ln (s+b)\Big]_s^\infty = \left[\ln \frac{s+a}{s+b}\right]_s^\infty$$

$$0 - \ln \frac{s+a}{s+b} = \ln \frac{s+b}{s+a}, \quad -a < \sigma. \qquad [162]$$

The derived pair, with $a < b$ is

$$\ln \frac{s+b}{s+a} \qquad \frac{e^{-at} - e^{-bt}}{t}, \qquad 0 \leqq t. \qquad [163]$$

18. THEOREM 18, INTEGRATION WITH RESPECT TO SECOND INDEPENDENT VARIABLE

If a *is a second variable independent of* t *and* s, *and* f(t,a) *is* $\mathfrak{L}$ *transformable with respect to* t *and has the* $\mathfrak{L}$ *transform* F(s,a), *and if the integrals appearing exist, then*

$$\mathfrak{L}_t\left[\int_{a_0}^{a} f(t,a)da\right] = \int_{a_0}^{a} F(s,a)da. \qquad [164]$$

This theorem states that integration with respect to a second independent variable is invariant under transformation from one domain to the other; or stated otherwise, $\mathfrak{L}$ transformation with respect to t and integration with respect to a are commutative operations.

Since integration is a limit process this theorem follows from Theorem 12. Start with the integral definition of the direct transformation

$$\int_0^\infty f(t,a)e^{-st}dt = F(s,a)$$

and integrate both members with respect to a. Thus

$$\int_{a_0}^{a}\int_0^\infty f(t,a)e^{-st}dtda = \int_{a_0}^{a} F(s,a)da. \qquad [165]$$

Since a is independent of t and s, the order of integration may be changed

and, if the integrals exist,

$$\int_0^\infty \int_{a_0}^{a} f(t,a)da \cdot e^{-st}dt = \int_{a_0}^{a} F(s,a)da, \qquad [166]$$

or

$$\mathfrak{L}\left[\int_{a_0}^{a} f(t,a)da\right] = \int_{a_0}^{a} F(s,a)da$$

as stated in the theorem.

Example 1. Use Theorem 18 to find the $\mathfrak{L}$ transform of $(\sin \beta t)/t$ starting with the pair

$$\frac{s}{s^2+\beta^2} \quad \Bigg| \quad \cos \beta t, \qquad 0 \leqq t.$$

Here the second independent variable is β. Applying Theorem 18 by integrating with respect to β between the limits 0 and β, the result is

$$\int^{\beta} \cos \beta t d\beta = \left[\frac{\sin \beta t}{t}\right]_0^{\beta} = \frac{\sin \beta t}{t}, \qquad 0 \leqq t, \qquad [167]$$

and

$$\int_0^{\beta} \frac{s}{s^2+\beta^2} d\beta = \left[\tan^{-1}\frac{\beta}{s}\right]_0^{\beta} = \tan^{-1}\frac{\beta}{s}, \qquad 0 < \sigma. \qquad [168]$$

Thus there is derived the pair

$$\tan^{-1}\frac{\beta}{s} \quad \Bigg| \quad \frac{\sin \beta t}{t}, \qquad 0 \leqq t. \qquad [169]$$

19. THEOREM 19, REAL MULTIPLICATION

If $f_1(t)$ *and* $f_2(t)$ *are* $\mathfrak{L}$*-transformable functions having the* $\mathfrak{L}$ *transforms* $F_1(s)$ *and* $F_2(s)$, *respectively, then*

$$\mathfrak{L}[f_1(t)f_2(t)] = \frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_1(s-w)F_2(w)dw, \qquad [170]$$

$$\max(\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_1}+\sigma_{a_2}) < \sigma, \qquad \sigma_{a_2} < c_2 < \sigma - \sigma_{a_1},$$

in which c_2 *is a real constant,* $\sigma = \Re[s]$, *and* σ_{a_1} *and* σ_{a_2} *are the abscissas of absolute convergence of functions* $f_1(t)$ *and* $f_2(t)$, *respectively.*

The process expressed by the integral will be called *convolution in the complex domain,* or more briefly *complex convolution,* and the functions $F_1(s)$ and $F_2(s)$ will be said to be convolved. The integral may be abbreviated to $F_1(s) \circledast F_2(s)$.

The theorem states that the $\mathfrak{L}$ transform of the product of two functions of the real variable is found by convolving the $\mathfrak{L}$ transforms of

these two functions. Thus multiplication in the real domain goes over into complex convolution in the complex domain [APPEN C, PI 4, ME 4].

This theorem is analogous to Theorem 9, complex multiplication. Complex convolution has many features in common with real convolution described in Sec. 2. The most important of these will be pointed out here.

In the line integral,

$$\frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_1(s-w)F_2(w)dw,$$

folding, translation, multiplication, and integration are indicated. In the complex w-plane the function $F_1(w)$ and hence the geometric pattern of its singularities and zeros is first folded about the axis of imaginaries and then translated by the complex variable s. Since the translations are limited to those which keep $\sigma_{a_2} < c_2 < \sigma - \sigma_{a_1}$, the path of integration from $c_2 - j\infty$ to $c_2 + j\infty$ lies in an analytic strip.

If $F(s)$ is the $\mathfrak{L}$ transform of $f_1(t)f_2(t)$, then the theorem follows from the integral definition of the direct transformation

$$F(s) = \int_0^\infty f_1(t)f_2(t)e^{-st}dt, \quad \max(\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_1} + \sigma_{a_2}) < \sigma,$$

by substituting for $f_2(t)$ its integral representation in terms of $F_2(s)$ given in Theorem 2, Sec. 14, Chapter 4.

$$F(s) = \int_0^\infty f_1(t)\cdot\frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_2(w)e^{tw}dw\cdot e^{-st}dt, \qquad [171]$$

$$\max(\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_1} + \sigma_{a_2}) < \sigma, \quad \sigma_{a_2} < c_2.$$

As indicated, the integration is to be carried out first with respect to the complex variable w and then with respect to the real variable t.

Since the functions are $\mathfrak{L}$ transformable it is permissible to change the order in which the two integrations are performed, with the result that

$$F(s) = \frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_2(w)\int_0^\infty f_1(t)e^{-(s-w)t}dtdw, \qquad [172]$$

with the same restrictions as in equation 171. But

$$\int_0^\infty f_1(t)e^{-(s-w)t}dt = F_1(s-w), \quad \begin{cases}\sigma_{a_1} < \sigma - \mathcal{R}[w], \\ \text{or } \mathcal{R}[w] < \sigma - \sigma_{a_1}.\end{cases} \qquad [173]$$

Since

$$F(s) = \mathfrak{L}[f_1(t)f_2(t)], \qquad [174]$$

then

$$\mathcal{L}[f_1(t)f_2(t)] = \frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_1(s-w)F_2(w)dw, \qquad [175]$$

$$\max(\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_1}+\sigma_{a_2}) < \sigma,\ \sigma_{a_2} < c_2 < \sigma - \sigma_{a_1},$$

as stated in the theorem.

Since the aim in the present volume is to keep the treatment as simple as possible, a discussion of the integration in the complex plane indicated in equation 175 will be postponed until Volume 2. This does not prevent, however, the use of the theorem in certain special but very useful forms which do not require this complex integration. Two of these forms are given below. The first, Theorem 19–*a*, applies if at least one transform factor has first-order poles only. The second, Theorem 19–*b*, applies if at least one transform factor has multiple-order poles.

THEOREM 19–*a*. *If* $f_1(t)$ *and* $f_2(t)$ *are* $\mathcal{L}$*-transformable functions having the* $\mathcal{L}$ *transforms* $F_1(s)$ *and* $F_2(s)$, *respectively, and if* $F_1(s) \triangleq \dfrac{A_1(s)}{B_1(s)}$ *is a rational algebraic fraction having* q *first-order poles and no others, then*

$$\mathcal{L}[f_1(t)f_2(t)] = \sum_{k=1}^{q}\frac{A_1(s_k)}{B_1'(s_k)}F_2(s-s_k). \qquad [176]$$

This particularly simple form can be derived from pair 1, Table 1, Chapter 6,

$$f_1(t) \triangleq \sum_{k=1}^{q}\frac{A_1(s_k)}{B_1'(s_k)}e^{s_kt}\ (=)\ \mathcal{L}^{-1}\left[\frac{A_1(s)}{B_1(s)}\right], \qquad 0 \leqq t, \qquad [177]$$

by multiplying the left member by $f_2(t)$ and finding the $\mathcal{L}$ transform of the product. Thus

$$\begin{aligned}\mathcal{L}[f_1(t)f_2(t)] &= \mathcal{L}\left[\sum_{k=1}^{q}\frac{A_1(s_k)}{B_1'(s_k)}e^{s_kt}f_2(t)\right]\\ &= \sum_{k=1}^{q}\frac{A_1(s_k)}{B_1'(s_k)}\mathcal{L}[e^{s_kt}f_2(t)]\\ &= \sum_{k=1}^{q}\frac{A_1(s_k)}{B_1'(s_k)}F_2(s-s_k). \qquad [178]\end{aligned}$$

The change in order of summation and $\mathcal{L}$ transformation is justified by the linearity theorem (Theorem 5); the replacement of $\mathcal{L}[e^{s_kt}f_2(t)]$ by $F_2(s-s_k)$ is made with the complex translation theorem (Theorem 11).

Example 1. Find the $\mathfrak{L}$ transform of $\sin \alpha t \cdot \sin \beta t$.

If $f_1(t) \triangleq \sin \alpha t$ and $f_2(t) \triangleq \sin \beta t$, then $F_1(s) = \dfrac{\alpha}{s^2 + \alpha^2}$ and $F_2(s) = \dfrac{\beta}{s^2 + \beta^2}$. The poles of $F_1(s)$ are at $\pm j\alpha$. Then by equation 176,

$$\mathfrak{L}[\sin \alpha t \cdot \sin \beta t] = \frac{\alpha}{2(j\alpha)} \cdot \frac{\beta}{(s - j\alpha)^2 + \beta^2} + \frac{\alpha}{2(-j\alpha)} \cdot \frac{\beta}{(s + j\alpha)^2 + \beta^2}$$

$$= \frac{2\alpha\beta s}{[s^2 + (\alpha + \beta)^2][s^2 + (\alpha - \beta)^2]}. \qquad [179]$$

That this is correct can be verified readily by recalling that $\sin \alpha t \cdot \sin \beta t \equiv \frac{1}{2}[\cos(\alpha - \beta)t - \cos(\alpha + \beta)t]$ and its $\mathfrak{L}$ transformation gives

$$\mathfrak{L}[\sin \alpha t \cdot \sin \beta t] = \frac{1}{2\; s^2 + (\alpha - \beta)^2} \quad 2\; s^2 + (\alpha + \beta)^2$$

$$\frac{2\alpha\beta s}{[s^2 + (\alpha - \beta)^2][s^2 + (\alpha + \beta)^2]}. \qquad [180]$$

Example 2. Find $F(s) \circledast F(s)$ if $F(s)$ is a rational algebraic fraction having q poles all of first order.

Let $F(s) \triangleq A(s)/B(s)$, then it follows from equation 178 that

$$F(s) \circledast F(s) = \sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} \cdot \frac{A(s - s_k)}{B(s - s_k)}. \qquad [181]$$

Note that by this process the $\mathfrak{L}$ transform of the square of one type of time function has been determined directly from the transform of the function.

THEOREM 19-b. *Let* $f_1(t)$ *and* $f_2(t)$ *be* $\mathfrak{L}$*-transformable functions having the* $\mathfrak{L}$ *transforms* $F_1(s)$ *and* $F_2(s)$, *respectively, and let* $F_1(s)$ *be a rational algebraic fraction having* n *distinct poles* $s_1, s_2, \cdots, s_n$ *with*

s_1 *of multiplicity* m_1
s_2 *of multiplicity* m_2

s_n *of multiplicity* m_n

subject to the restriction $m_1 + m_2 + \cdots + m_n = q$. *Then*

$$\mathfrak{L}[f_1(t)f_2(t)] = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{(-1)^{m_k - j} K_{kj}}{(m_k - j)!} \left[\frac{d^{m_k - j}}{ds^{m_k - j}} F_2(s) \right]_{s = s - s_k} \qquad [182]$$

in which

$$K_{kj} \triangleq \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} (s - s_k)^{m_k} F_1(s) \right]_{s = s_k}. \qquad [183]$$

This special form is based upon the partial-fraction expansion of a rational algebraic transform having multiple-order poles (see pair 2, Table 1, Chapter 6). Since $F_1(s)$ has multiple-order poles, its $\mathfrak{L}$ transform is

$$f_1(t) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{K_{kj}}{(m_k - j)!} t^{m_k - j} e^{s_k t}, \tag{184}$$

in which K_{kj} is defined in equation 183. If equation 184 is multiplied by $f_2(t)$ and the product is $\mathfrak{L}$ transformed,

$$\begin{aligned} \mathfrak{L}[f_1(t)f_2(t)] &= \mathfrak{L}\left[\sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{K_{kj}}{(m_k - j)!} t^{m_k - j} e^{s_k t} f_2(t)\right] \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{K_{kj}}{(m_k - j)!} \mathfrak{L}[t^{m_k - j} e^{s_k t} f_2(t)]. \end{aligned} \tag{185}$$

The change in order of double summation and $\mathfrak{L}$ transformation is justified by the linearity theorem. From the complex differentiation theorem (Theorem 16),

$$\mathfrak{L}[t^{m_k - j} f_2(t)] = (-1)^{m_k - j} \frac{d^{m_k - j}}{ds^{m_k - j}} F_2(s). \tag{186}$$

From the complex translation theorem (Theorem 11),

$$\mathfrak{L}[e^{s_k t} t^{m_k - j} f_2(t)] = (-1)^{m_k - j} \left[\frac{d^{m_k - j}}{ds^{m_k - j}} F_2(s)\right]_{s = s - s_k}. \tag{187}$$

Substituting from equation 187 in 185,

$$\mathfrak{L}[f_1(t)f_2(t)] = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{(-1)^{m_k - j} K_{kj}}{(m_k - j)!} \left[\frac{d^{m_k - j}}{ds^{m_k - j}} F_2(s)\right]_{s = s - s_k} \tag{188}$$

as stated in equation 182.

Example 1. If the $\mathfrak{L}$ transform of $f(t)$ is $\dfrac{s}{(s+\gamma)(s+\alpha)^2}$ find the $\mathfrak{L}$ transform of $f^2(t)$ directly by complex convolution.

Here $n = 2$. Let $s_1 \triangleq -\gamma$ and $s_2 \triangleq -\alpha$; then $m_1 = 1$ and $m_2 = 2$, and

$$\mathfrak{L}[f(t)] = \frac{s}{(s+\gamma)(s+\alpha)^2} = \frac{K_1}{s+\gamma} + \frac{K_{21}}{(s+\alpha)^2} + \frac{K_{22}}{s+\alpha} \tag{189}$$

in which

$$\begin{aligned} K_1 &\triangleq \left[\frac{s}{(s+\alpha)^2}\right]_{s=-\gamma} = \frac{-\gamma}{(\alpha-\gamma)^2} \\ K_{21} &\triangleq \left[\frac{s}{s+\gamma}\right]_{s=-\alpha} = \frac{\alpha}{\alpha-\gamma} \\ K_{22} &\triangleq \left[\frac{d}{ds}\frac{s}{s+\gamma}\right]_{s=-\alpha} = \frac{\gamma}{(\alpha-\gamma)^2}. \end{aligned}$$

Finally, by equation 182,

$$\mathfrak{L}[f^2(t)] = \left[\frac{K_1}{s+\gamma} + \frac{K_{21}}{(s+\alpha)^2} + \frac{K_{22}}{s+\alpha}\right] \quad (s+\gamma)(s+\alpha)^2$$

$$= K_1\left[\frac{s}{(s+\gamma)(s+\alpha)^2}\right]_{s=s+\gamma} - K_{21}\left[\frac{d}{ds}\,\frac{s}{(s+\gamma)(s+\alpha)^2}\right]_{s=s+\alpha}$$

$$+ K_{22}\left[\frac{s}{(s+\gamma)(s+\alpha)^2}\right]_{s=s+\alpha}$$

$$= \frac{K_1(s+\gamma)}{(s+2\gamma)(s+\alpha+\gamma)^2} + \frac{K_{21}[2(s+\alpha)^2+\gamma s]}{(s+\alpha+\gamma)^2(s+2\alpha)^3}$$

$$+ \frac{K_{22}(s+\alpha)}{(s+\alpha+\gamma)(s+2\alpha)^2}. \qquad [190]$$

It is seen that despite the rather formidable appearance of equation 182 the procedure that it prescribes is not difficult to understand or to carry out.

20. THEOREM 20, COMMUTATIVITY OF $\mathfrak{L}$ WITH $\mathfrak{R}$ AND $\mathfrak{I}$ TRANSFORMATIONS

If the complex function f(t) *is* $\mathfrak{L}$ *transformable and has the* $\mathfrak{L}$ *transform* F(s), *then*

$$\left.\begin{aligned} &(a)\ \mathfrak{L}\{\mathfrak{R}[f(t)]\} = \mathfrak{R}\{\mathfrak{L}[f(t)]\} = \mathfrak{R}[F(s)],\\ &(b)\ \mathfrak{L}\{\mathfrak{I}[f(t)]\} = \mathfrak{I}\{\mathfrak{L}[f(t)]\} = \mathfrak{I}[F(s)].\end{aligned}\right\} \qquad [191]$$

This theorem states that the $\mathfrak{L}$ transformation is commutative with the $\mathfrak{R}$ and $\mathfrak{I}$ transformations. The demonstration of this follows directly from the linear property of the integral defining the $\mathfrak{L}$ transformation. Let $p(t)$ and $q(t)$ be $\mathfrak{L}$-transformable real functions with $\mathfrak{L}$ transforms $P(s)$ and $Q(s)$, respectively. In the integral definition of the direct transformation,

$$\int_0^\infty f(t)e^{-st}ds = F(s),$$

let $f(t) \triangleq p(t) + jq(t)$. Then by the linearity theorem,

$$\int_0^\infty [p(t) + jq(t)]e^{-st}dt = \int_0^\infty p(t)e^{-st}dt + j\int_0^\infty q(t)e^{-st}dt$$

$$= P(s) + jQ(s) = F(s). \qquad [192]$$

That is,

$$\mathfrak{L}\{\mathfrak{R}[f(t)]\} + j\mathfrak{L}\{\mathfrak{I}[f(t)]\} = \mathfrak{R}\{\mathfrak{L}[f(t)]\} + j\mathfrak{I}\{\mathfrak{L}[f(t)]\}, \qquad [193]$$

and since in a complex equation the real parts must be equal and the imaginary parts must be equal separately,

$$\left.\begin{aligned}(a)\quad &\mathfrak{L}\{\mathcal{R}[f(t)]\} = \mathcal{R}\{\mathfrak{L}[f(t)]\},\\ (b)\quad &\mathfrak{L}\{\mathcal{I}[f(t)]\} = \mathcal{I}\{\mathfrak{L}[f(t)]\},\end{aligned}\right\} \qquad [194]$$

as stated in the theorem.

Example 1. The complex function $e^{j\beta t} = \cos \beta t + j \sin \beta t$. Then

$$\mathfrak{L}[e^{j\beta t}] = \mathfrak{L}[\cos \beta t] + j\mathfrak{L}[\sin \beta t]$$

$$= \frac{s}{s^2+\beta^2} + j\frac{\beta}{s^2+\beta^2} = \frac{1}{s - j\beta}, \qquad [195]$$

and it is seen that

$$\left.\begin{aligned}(a)\quad &\mathfrak{L}\{\mathcal{R}[e^{j\beta t}]\} = \frac{s}{s^2+\beta^2} = \mathcal{R}\left[\frac{1}{s-j\beta}\right] = \mathcal{R}\{\mathfrak{L}[e^{j\beta t}]\},\\ (b)\quad &\mathfrak{L}\{\mathcal{I}[e^{j\beta t}]\} = \frac{\beta}{s^2+\beta^2} = \mathcal{I}\left[\frac{1}{s-j\beta}\right] = \mathcal{I}\{\mathfrak{L}[e^{j\beta t}]\}.\end{aligned}\right\} \qquad [196]$$

PROBLEMS

8·1. The $\mathfrak{L}$ transform of the current that results from application of a unit step voltage in a certain network has the form

$$\frac{a_2s^2 + a_1s + a_0}{s[(s+\alpha)^2 + \beta^2]}.$$

If the time constant of the network is to be taken as the unit of time, use Theorem 8 to find the time function for the current. Let $\lambda \triangleq 2\pi$ (time constant/period).

8·2. Apply Theorem 9 in finding the $\mathfrak{L}^{-1}$ transform of

$$\frac{1}{[(s+\alpha)^2+\beta^2](s+\gamma)}.$$

8·3. A system whose initial energy storage is zero is subjected to a unit step driving force. Its response is

$$Ae^{-\alpha t}\sin(\beta t + \psi), \qquad 0 \leqq t.$$

Find the response of this system, under the same initial conditions, to a driving force of the form $Be^{-\sigma t}$.

8·4. Solve the convolution-type integral equation

$$x(t) + \int_0^t (t-\tau)x(\tau)d\tau = 1, \qquad 0 \leqq t,$$

for the unknown function $x(t)$ by use of the $\mathfrak{L}$ transformation.

8·5. Using the $\mathfrak{L}$ transformation, solve the convolution-type integral equation

$$[a(t) * f(t) * f(t)] - [b(t) * f(t)] + c(t) = 0, \qquad 0 \leqq t,$$

in which

$$a(t) \triangleq \cos \beta t, \qquad b(t) \triangleq \sin \beta t, \qquad \text{and} \qquad c(t) = \tfrac{1}{4}(1 - \cos \beta t).$$

8-6. Show that if $\mathfrak{L}[f(t)] \triangleq F(s)$, then

$$\mathfrak{L}[f(t)u(t-a)] = F(s) - \int_0^a f(t)e^{-st}dt.$$

8-7. If $\mathfrak{L}[f(t)] \triangleq F(s)$, give the $\mathfrak{L}$ transforms of the following functions:

(a) $f(t)[u(t-a) - u(t-b)]$.

(b) $f(t)[u(b-t) - u(a-t)]$.

(c) $f(t)u(a-t)$.

(d) $f(t)u(t-a)u(b-t)$.

Assume that a and b are positive real numbers and $a < b$.

8-8. Use Theorem 10 to find the transform of the section between $t = a$ and $t = b$ of a cosine curve, i.e., find the transform of

$$\cos \beta t \cdot [u(t-a) - u(t-b)], \qquad a < b.$$

8-9. Sketch the $\mathfrak{L}^{-1}$ transforms of the following functions:

$$(a)\ \frac{1-e^{-s}}{s}, \qquad (b)\ \left(\frac{1-e^{-s}}{s}\right)^2, \qquad (c)\ \left(\frac{1-e^{-s}}{s}\right)^3.$$

8-10. Find the $\mathfrak{L}^{-1}$ transform of $\dfrac{1-e^{-s}}{s^2(1+e^{-s})}$ by means of an expansion (a) in partial fractions, and (b) in a series of exponentials. Sketch the time function represented.

8-11. Using Theorem 10, show that if $A(s)/B(s)$ is a rational algebraic fraction having q poles, none at the origin and all of the first order, and if a is a positive real number, then

$$\mathfrak{L}^{-1}\left[\frac{A(s)}{B(s)}(1-e^{-as})\frac{1}{s}\right] (=) \begin{cases} \dfrac{A(0)}{B(0)} + \displaystyle\sum_{k=1}^{q} \frac{A(s_k)}{s_kB'(s_k)} e^{s_kt}, & 0 \leqq t \leqq a, \\ \displaystyle\sum_{k=1}^{q} \frac{A(s_k)(1-e^{-s_ka})}{s_kB'(s_k)} e^{s_kt}, & a \leqq t. \end{cases}$$

NOTE: This shows that $(1 - e^{-as})$ in the numerator of the transform may be treated as an algebraic factor in finding the $\mathfrak{L}^{-1}$ transform for $a < t$.

8-12. If the $\mathfrak{L}$ transform of the section between a and b of a function is $\Phi(s)$, what is the transform of the new function produced by turning this section end for end on the time axis but still keeping it between a and b?

8-13. Using Theorem 11, find the $\mathfrak{L}^{-1}$ transform of

$$\frac{(s+\alpha)^2 + \lambda^2}{[(s+\alpha)^2 + \beta^2](s+\alpha)}.$$

8·14. Use Theorem 11 to find the $\mathfrak{L}$ transform of a sine wave which is "amplitude" modulated by a sawtooth wave, i.e., find the transform of

$$\left[t - \sum_{k=1}^{\infty} u(t-k)\right] \sin \lambda t,$$

in which k and $\lambda/2\pi$ are positive integers.

8·15. If the response of a system, initially without stored energy, to a unit impulse disturbance is

$$\frac{1}{\sin\theta} e^{-\alpha t} \sin(\beta t + \theta), \qquad 0 \leqq t,$$

in which $\theta \triangleq \tan^{-1}\beta/b$, and b is a real constant, find the response of the system, under the same initial conditions, to a disturbance $(1/\lambda)\sin \lambda t$.

8·16. Show that the response of a system to a rectangular pulse whose duration is short compared with the smallest time constant of the system approximates the response of this system to an impulse. The area under the pulse and the magnitude of the impulse are to be taken equal.

8·17. Starting with the pair

$$\frac{1}{[(s+\alpha)^2+\beta^2](s+\alpha)} \quad \Bigg| \quad \frac{1}{\beta^2} e^{-\alpha t}(1 - \cos \beta t), \qquad 0 \leqq t,$$

use Theorem 12 to find the $\mathfrak{L}^{-1}$ transform of $(s+\alpha)^{-3}$.

8·18. Apply Theorem 13 to $[(s+\alpha)^2+\beta^2]^{-1}$ and its $\mathfrak{L}^{-1}$ transform to find the $\mathfrak{L}^{-1}$ transforms of the following functions:

$$(a)\ \frac{s+\alpha}{[(s+\alpha)^2+\beta^2]^2}, \qquad (b)\ \frac{1}{[(s+\alpha)^2+\beta^2]^2}.$$

8·19. Since the inverse-transformation process presents the major difficulty in the use of the $\mathfrak{L}$-transformation method, it is worth while to know how to gain certain useful information about the inverse transform from inspection of the transform. For example, the transform for the output voltage $v_2(t)$ of a certain 2-section lattice-type network is

$$\frac{a_3s^3 + a_2s^2 + a_1s + a_0}{s(s+\gamma)^2[(s+\alpha)^2+\beta^2]}$$

in which a_3, a_2, a_1, a_0, α, β, and γ are positive real numbers assumed to be known. Without actually carrying out an $\mathfrak{L}^{-1}$ transformation, give:

(a) The form of each term in $v_2(t)$.
(b) The initial value $v_2(0)$.
(c) The initial value of the first derivative $v_2'(0)$.
(d) The value which $v_2(t)$ will approach for large values of t.

8·20. A certain system has a driving-point function in the form of a rational algebraic fraction with poles and zeros as follows:

First-order poles at $-\alpha \pm j\beta$,
Second-order pole at $-\delta$,
First-order zeros at $-\gamma$, $-\mu$.

The coefficient of the fraction is A. Here α, β, δ, γ, μ, and A are positive real numbers, all different.

(1) If a driving force in the form of a unit step function is applied to this system, give, for the driving-point response (*a*) the initial value, (*b*) the initial slope, (*c*) the final value, and (*d*) the general form of the functions present.

(2) If a driving force in the form of a unit sine wave of angular frequency ω_1 is applied to this system, give the steady-state part of the driving-point response. Here $\omega_1 \neq \beta$.

8·21. If the transform of $f(t)$ is $(a_1s + a_0)/(s^3 + b_2s^2 + b_1s + b_0)$, use Theorems 14 and 15 to find (*a*) the value of $f(t)$ as $t \to \infty$, and (*b*) the values of $f(t)$ and its first three derivatives at $t = 0+$. It may be assumed that all poles lie to the left of the axis of imaginaries.

8·22. Starting with the pair stated in problem 8·17, use Theorems 6–*a* and 7–*a* to find the $\mathfrak{L}^{-1}$ transforms, respectively, of

$$(a)\ \frac{s}{[(s+\alpha)^2+\beta^2](s+\alpha)}, \qquad (b)\ \frac{1}{s[(s+\alpha)^2+\beta^2](s+\alpha)}.$$

8·23. Starting with the function $e^{-\alpha t}\cos^2 \beta t$ and its $\mathfrak{L}$ transform, find the $\mathfrak{L}$ transform of $te^{-\alpha t}\cos^2 \beta t$ by use of Theorem 16.

8·24. Using Theorem 17, find $\mathfrak{L}\left[\dfrac{\sin^2 \beta t}{t}\right]$.

8·25. Evaluate the integral $\displaystyle\int_0^\alpha \frac{d\alpha}{(\alpha+s)^2+\beta^2}$ by use of Theorem 18.

8·26. (*a*) Find the $\mathfrak{L}$ transform of the product $t^2e^{-\alpha t}\cdot \sin \beta t$ by use of Theorem 19–*a*.

(*b*) Repeat, using Theorem 19–*b*.

8·27. Find the $\mathfrak{L}$ transform of the product of the functions $(t+1)^2e^{-\alpha t}$ and $\sin \beta t$ by convolving their respective transforms.

8·28. In a certain automatic-control system the $\mathfrak{L}$ transform for the error following a disturbance has the form $\dfrac{a_1s + a_0}{(s+\alpha)^2+\beta^2}$. From this transform find directly the transform of the square of the error.

8·29. (*a*) Using Theorem 10 find the $\mathfrak{L}$ transform of the vacuum-tube plate current described in problem 5·6.

(*b*) Find the fundamental and first harmonic of the steady-state voltage that this current produces across the tuned circuit, basing the calculations upon the $\mathfrak{L}$ transform of this voltage.

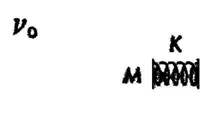


FIG. 8·P30

8·30. If it is assumed that the "relative approach" (flattening) of a sphere when pressed against a plane is proportional to the first power of the contact compressive force (probably it is more nearly proportional to the ⅔ power) then the following integral equation can be written for this force $f(t)$ when the sphere of mass m (see diagram) strikes with velocity v_0 a mass M whose displacement is elastically constrained by a spring of stiffness K:

$$af(t) = v_0t - \frac{1}{m}\int_0^t f(\tau)(t-\tau)d\tau - \frac{1}{m\gamma}\int_0^t f(\tau)\sin \gamma(t-\tau)d\tau, \qquad 0 \leqq t \leqq t_1.$$

Here a is the relative approach per unit force, $\gamma \triangleq \sqrt{K/M}$, and t_1 is the total time of contact.

In the right member of the integral equation the first term is the forward displacement that m would experience if it were a free body. The second term is the backward displacement of m caused by the compressive force. The difference between these two terms is the net displacement of the center of m while in contact with M. The third term is the displacement forward of M caused by the compressive force while m and M are in contact. The difference between the net displacement of the center of m and the displacement of M is the relative approach.

(*a*) Derive each of the three terms in the right member of the integral equation.

(*b*) By means of the $\mathfrak{L}$ transformation find the compressive force during the interval of contact.

8·31. In geophysical prospecting by means of electric transients it is desired to obtain the transient voltage between two probes stuck in the earth when a unit impulse of current $u_1(t)$ is applied between two contacts or grounds in the neighborhood of the probes. Since physically only an approximation to a current impulse can be produced, there arises the problem of deriving from the response to this approximate impulse the response to the mathematical impulse $u_1(t)$.

If the approximate current impulse is produced by a portable surge generator which, when connected to the contacts, gives a surge of the form $i(t) = Ae^{-at}$, and if the recorded response between the potential probes is $v(t)$, how would you correct this experimentally determined $v(t)$ to obtain the response of the system to a mathematical unit impulse? For systems of this type $v(0) = 0$.

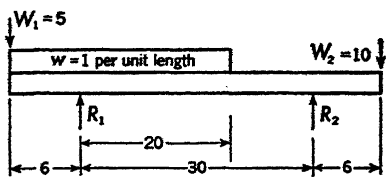


FIG. 8·P32

8·32. Find the static deflection curve of the beam shown in the diagram. The beam overhangs its supports and bears a uniformly distributed load over a portion of its length and concentrated loads at its two ends. The dimensions and weights, in a single system of units, have magnitudes as indicated. The weight of the beam may be neglected.

Chapter IX

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

In the previous chapters the application of the $\mathfrak{L}$-transformation method to the solution of linear constant-coefficient integrodifferential equations has been treated. In the present chapter[1] it will be shown how this transformation method can be used to solve linear constant-coefficient difference equations and also combined integrodifferential and difference equations, i.e., i-d difference equations [He 3, Jo 2, Ne 1]. It will serve in the completed treatment as a logical transition chapter between one-dimensional problems, expressible by integrodifferential equations, and multi-dimensional problems, expressible by partial integrodifferential equations (Volume 2).

Difference equations arise in systems in which the variables change by finite amounts as in statistics and in interpolation theory, or by quanta as in atomic physics. They also find application in physical systems in which there is a recurrence of structure and in systems in which cyclic switching takes place. For example, in finance they are applicable to problems concerning interest, sinking funds, amortization, annuities, mortgages, and loan retirements. In electric systems they are applicable to artificial lines, wave filters, lump-loaded telephone lines, network representations of transformer and machine windings, multi-stage amplifiers, surge generators, suspension insulator strings, and potential dividers. They are applicable also in networks in which there is cyclic switching produced for example by mechanical commutators or by electronic switching devices. In mechanical systems they are applicable to periodically loaded strings, mechanical filters, continuous beams with equidistant points of support, lattice trusses, and crankshafts of multicylinder engines. In all of these examples it will be noted that there is a repetition either of equal intervals of time such as interest periods or switching cycles or of equal units of structure such as sections in artificial lines.

[1] Certain of the points presented in this chapter were developed in the Master's thesis of D. F. Tuttle, Jr., written under the guidance of one of the authors at the Mass. Inst. of Tech. in 1938.

Difference equations can be recognized by the presence of finite differences in the successive values of the independent variable. Their solutions involve discontinuities, i.e., jumps, at equally spaced values of the independent variable.

Because of the essential discontinuity of these solutions, it is necessary to present first certain elements basic to the $\mathfrak{L}$ transformation of functions having regularly spaced discontinuities. These elements are then applied to find the direct transforms of a number of discontinuous functions, thereby building up a table of useful transform pairs. Following this the $\mathfrak{L}^{-1}$ transformation is presented as the inverse of the direct transformation. Finally, in a few examples, difference equations are set up for physical systems and solved by means of the $\mathfrak{L}$-transformation method, attention being directed mainly to problems in which there is a transient.

A. BASIC THEORY

1. Jump Functions

A special type of step function which changes value only at integral values of x and hence remains constant in the interval between steps will be called a *jump function.* It will be denoted by a symbol $\int$ in front of the usual function symbol such as $y(x)$. Thus jump $y(x)$ is denoted by $\int y(x)$. Of course the jumps may be either increases or decreases in the function. In using a jump function the value at a discontinuity will be taken as the value of the function as the argument approaches the point of discontinuity from the right. Thus if $\int y(x)$ has a discontinuity at $x = a$, the value of $\int y(a) \triangleq \int y(a+)$. The + sign will usually be omitted for brevity.

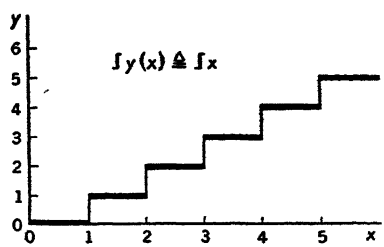


FIG. 9·1

A function that will be found to be basic in the representation of jump functions is the unit pulse, $p(x) \triangleq u(x) - u(x-1) = u(x)u(1-x)$. If r is an integer, the shifted unit pulse is $p(x-r) \triangleq u(x-r) - u(x-r-1)$.

In terms of pulses, $\int y(x)$ with steps at intervals of unity can be expressed as

$$\int y(x) = \sum_{r=0}^{\infty} y(r)p(x-r). \qquad [1]$$

For example, if $\mathcal{J}y(x) \triangleq \mathcal{J}x$, for $0 \leq x$, then

$$\begin{aligned}\mathcal{J}x &= p(x-1) + 2p(x-2) + \cdots + rp(x-r) + \cdots \\ &= \sum_{r=1}^{\infty} rp(x-r).\end{aligned} \quad [2]$$

This jump function (Fig. 9·1) has the form of a staircase with unit risers at the integral points. It is the jump function that interpolates the points of an infinite arithmetic series of numbers.

2. General Forms of the Linear Second-Order Difference Equation with Constant Coefficients

A general form of linear second-order difference equation with constant coefficients [Bo 4, Mi 1, No 1] is

$$a_0\mathcal{J}y(x) + a_1\mathcal{J}y(x+1) + a_2\mathcal{J}y(x+2) = \mathcal{J}f(x). \quad [3]$$

In this equation $\mathcal{J}y(x)$ is the unknown function, $\mathcal{J}y$ being the dependent variable and x the independent variable. The constant coefficients a_r and the driving function $\mathcal{J}f(x)$ are known. The equation states the relation holding among values of $\mathcal{J}y(x)$ for values of x separated by unity over a range of 2 units. There is no loss of generality by this choice of unit length, because an equation in which the increments of x are of length h can be converted into an equation in which the increments are of unit length by making the change of variable $x = h\tilde{x}$.

The form of the difference equation 3 is especially convenient for application to physical problems. However the equation can be put in another form which displays more clearly its analogy to the differential equation.

Let the *first difference* be

$$\Delta\mathcal{J}y(x) \triangleq \mathcal{J}y(x+1) - \mathcal{J}y(x). \quad [4]$$

Successive applications of this definition give higher-order differences [Hu 4]. For example, the second- and third-order differences are

$$\left.\begin{aligned}\Delta^2\mathcal{J}y(x) &= \Delta[\Delta\mathcal{J}y(x)] = \Delta[\mathcal{J}y(x+1) - \mathcal{J}y(x)] \\ &= \mathcal{J}y(x+2) - 2\mathcal{J}y(x+1) + \mathcal{J}y(x), \\ \Delta^3\mathcal{J}y(x) &= \Delta[\Delta^2\mathcal{J}y(x)] = \Delta[\mathcal{J}y(x+2) - 2\mathcal{J}y(x+1) + \mathcal{J}y(x)] \\ &= \mathcal{J}y(x+3) - 3\mathcal{J}y(x+2) + 3\mathcal{J}y(x+1) - \mathcal{J}y(x).\end{aligned}\right\} \quad [5]$$

From equation 4

$$\mathcal{J}y(x+1) = (1+\Delta)\mathcal{J}y(x). \quad [6]$$

Similarly, equations 5 yield

$$\left.\begin{aligned} \smallint y(x+2) &= (1+\Delta)\smallint y(x+1) = (1+\Delta)^2 \smallint y(x), \\ \smallint y(x+3) &= (1+\Delta)\smallint y(x+2) = (1+\Delta)^3 \smallint y(x). \end{aligned}\right\} \quad [7]$$

If results 6 and 7 are substituted in equation 3, it takes the form

$$b_2\Delta^2 \smallint y(x) + b_1 \Delta \smallint y(x) + b_0 \smallint y(x) = \smallint f(x). \quad [8]$$

In terms of the a_r coefficients, the b_r coefficients are

$$\left.\begin{aligned} b_0 &\triangleq a_0 + a_1 + a_2, \\ b_1 &\triangleq a_1 + 2a_2, \\ b_2 &\triangleq a_2. \end{aligned}\right\} \quad [9]$$

The analogy between equation 8 and the second-order linear differential equation with constant coefficients is readily apparent, Δ^r being analogous to $\dfrac{d^r}{dx^r}$. The basic dissimilarity between a difference equation and a differential equation resides in the lengths of the increments of the independent variable. These increments for the general difference equation are of length h, whereas those for the differential equation have approached length zero. Thus a differential equation is obtained from a difference equation by dividing each term by the same power of h as the order of its difference and then letting h approach zero.

As a preliminary to the discussion that follows, it will be convenient to introduce a simple symbol for the $\mathfrak{L}$ transform of the unit pulse:

$$\mathfrak{L}[p(x)] = \int_0^\infty p(x)e^{-sx}dx = \int_0^1 e^{-sx}dx = \frac{1-e^{-s}}{s} \triangleq P(s). \quad [10]$$

Then for the shifted unit pulse,

$$\mathfrak{L}[p(x-r)] = \int_r^{r+1} e^{-sx}dx = \frac{e^{-rs}(1-e^{-s})}{s} \triangleq e^{-rs}P(s). \quad [11]$$

3. THEOREM 21, TRANSLATION OF JUMP FUNCTIONS

If difference equations are to be solved by the $\mathfrak{L}$ transformation it is necessary to know the form taken by $\mathfrak{L}[\smallint y(x+1)]$ and $\mathfrak{L}[\Delta \smallint y(x)]$ in terms of $\mathfrak{L}[\smallint y(x)]$. As a basis for this the following theorem is developed; it is a modification of Theorem 10 (Sec. 5, Chapter 8).

If $\smallint y(x)$ *is an* $\mathfrak{L}$*-transformable jump function with the* $\mathfrak{L}$ *transform* $Y(s)$ *then*

$$\mathfrak{L}[\smallint y(x+1)] = e^s[Y(s) - y(0)P(s)], \quad [12]$$

in which P(s) *denotes the* $\mathfrak{L}$ *transform of the unit pulse* $p(x) \triangleq u(x) - u(x-1)$.

This theorem states that the $\mathfrak{L}$ transform of a jump function after its translation to the left by unity can be found from the $\mathfrak{L}$ transform of the original function by (1) subtracting the transform of that part of the original function in the range 0 to 1, and (2) multiplying the remainder by e^s. The part subtracted will be called the *partial transform* of the function. It is seen that a knowledge of $\mathit{\Gamma} y(x)$ over the range $0 \leqq x \leqq 1$ is required. Since $\mathit{\Gamma} y(x)$ is a jump function its partial transform is the transform of a pulse.

The theorem follows from the integral definition of the $\mathfrak{L}$ transformation,

$$\int_0^\infty \mathit{\Gamma} y(x) e^{-sx} dx = Y(s),$$

by first dividing the range of integration as follows,

$$\int_0^1 \mathit{\Gamma} y(x) e^{-sx} dx + \int_1^\infty \mathit{\Gamma} y(x) e^{-sx} dx = Y(s). \qquad [13]$$

If in the second integral of equation 13 the change of variable $x \triangleq \xi + 1$ is made, 13 becomes

$$\int_0^1 \mathit{\Gamma} y(x) e^{-sx} dx + e^{-s} \int_0^\infty \mathit{\Gamma} y(\xi + 1) e^{-s\xi} d\xi = Y(s). \qquad [14]$$

Since ξ is only a variable of integration it may be replaced by x and equation 14 can be put in the form

$$\int_0^\infty \mathit{\Gamma} y(x+1) e^{-sx} dx = e^s \left[Y(s) - \int_0^1 \mathit{\Gamma} y(x) e^{-sx} dx \right]. \qquad [15]$$

But the partial transform

$$\int_0^1 \mathit{\Gamma} y(x) e^{-sx} dx = y(0) \int_0^1 e^{-sx} dx \triangleq y(0) P(s). \qquad [16]$$

Thus equation 15 becomes

$$\mathfrak{L}[\mathit{\Gamma} y(x+1)] = e^s [Y(s) - y(0) P(s)],$$

as stated in the theorem.

By repeated applications of this theorem,

$$\left.\begin{aligned} &\mathfrak{L}[\mathit{\Gamma} y(x+2)] = e^{2s}[Y(s) - y(0)P(s) - y(1)e^{-s}P(s)], \\ &\mathfrak{L}[\mathit{\Gamma} y(x+3)] \\ &\qquad = e^{3s}[Y(s) - y(0)P(s) - y(1)e^{-s}P(s) - y(2)e^{-2s}P(s)], \end{aligned}\right\} \qquad [17]$$

and the procedure is similar for jump functions translated a greater number of units.

4. TRANSFORMS OF DIFFERENCES

With the aid of Theorem 21 the $\mathfrak{L}$ transforms of differences can now be written. Because of the nature of $\int y(x)$ it can be written as a sequence of shifted pulses as

$$\int y(x) = y(0)p(x) + y(1)p(x-1) + y(2)p(x-2) + \cdots. \qquad [18]$$

If $\mathfrak{L}[\int y(x)] = Y(s)$, the transform of the first difference is

$$\begin{aligned} \mathfrak{L}[\Delta \int y(x)] &= \mathfrak{L}[\int y(x+1) - \int y(x)] \\ &= e^s[Y(s) - y(0)P(s)] - Y(s) \\ &= (e^s - 1)Y(s) - y(0)e^sP(s). \end{aligned} \qquad [19]$$

Thus the transform of the first difference of a jump function is formed by (1) multiplying the transform of the function by $(e^s - 1)$ and (2) subtracting the product of e^s and the transform of the first pulse in the pulse sequence 18 composing the function.

Using the result given in equation 19, the transform of the second difference is

$$\begin{aligned} \mathfrak{L}[\Delta^2 \int y(x)] &= (e^s - 1)\mathfrak{L}[\Delta \int y(x)] - \Delta y(0)e^sP(s) \\ &= (e^s - 1)[(e^s - 1)Y(s) - y(0)e^sP(s)] - \Delta y(0)e^sP(s) \\ &= (e^s - 1)^2Y(s) - y(0)(e^s - 1)e^sP(s) - \Delta y(0)e^sP(s), \end{aligned} \qquad [20]$$

in which $\Delta y(0) = y(1) - y(0)$. Similarly, using the result given in equation 20, the transform of the third difference is

$$\begin{aligned} \mathfrak{L}[\Delta^3 \int y(x)] &= (e^s - 1)\mathfrak{L}[\Delta^2 \int y(x)] - \Delta^2 y(0)e^sP(s) \\ &= (e^s - 1)^3Y(s) - y(0)(e^s - 1)^2e^sP(s) \\ &\qquad - \Delta y(0)(e^s - 1)e^sP(s) \\ &\qquad - \Delta^2 y(0)e^sP(s), \end{aligned} \qquad [21]$$

in which $\Delta^2 y(0) = y(2) - 2y(1) + y(0)$.

5. ANALOGY BETWEEN TRANSFORMS OF DIFFERENCES AND OF DERIVATIVES

A comparison of the transforms of the *differences* of the jump function $y(x)$ and the transforms of the derivatives of the differentiable function $y_d(x)$ shows the close analogy that holds between them. Reading $\sim$ as "analogous to,"

$$\left.\begin{aligned} e^s - 1 &\sim s \\ y(0)e^sP(s) &\sim y_d(0) \\ \Delta y(0)e^sP(s) &\sim y_d'(0) \\ \Delta^2 y(0)e^sP(s) &\sim y_d''(0). \end{aligned}\right\} \qquad [22]$$

Based on the correspondence displayed in these analogies it can be anticipated that in the solution of *difference* equations the factor $(e^s - 1)$ will play the same role as s does in *differential* equations. Similarly in difference equations, the products of e^s and transforms of differences prescribed in the boundary region 0 to 1 will play the same roles as initial values of derivatives do in differential equations.

Since, from equations 4 and 5, the differences in the first interval are

$$\left.\begin{aligned} \Delta y(0) &= y(1) - y(0) \\ \Delta^2 y(0) &= y(2) - 2y(1) + y(0) \\ \Delta^3 y(0) &= y(3) - 3y(2) + 3y(1) - y(0), \end{aligned}\right\} \qquad [23]$$

it is seen that the boundary conditions for a kth-order difference equation can be given in either of two ways: (1) as the values of $\Gamma y(x)$ at the origin and at the succeeding $k - 1$ integral points or (2) as the values of the function and its first $k - 1$ differences in the interval $0 \leqq x \leqq 1$. The boundary conditions of a kth-order difference equation should be compared with those of a kth-order differential equation having for its boundary conditions the values at the origin of $y_d(x)$ and its first $k - 1$ derivatives.

6. General Procedure for Solving Difference Equations by $\mathfrak{L}$ Transformation

The general procedure for solving a linear constant-coefficient difference equation by use of the $\mathfrak{L}$ transformation can now be formulated. (1) The equation is transformed, introducing thereby the known boundary conditions as magnitudes of pulses. (2) The resulting equation is solved algebraically for the transform of the unknown function. (3) The inverse transformation is performed to obtain the desired solution.

Since a difference equation can be expressed in either of two ways, making the subsequent equations in the solution somewhat different, both procedures will be illustrated. Each will be applied to a second-order difference equation. Extensions to the kth-order equation will be evident.

(*a*) Starting with the difference equation in the *ordinate* form

$$a_0 \Gamma y(x) + a_1 \Gamma y(x + 1) + a_2 \Gamma y(x + 2) = \Gamma f(x), \qquad [3]$$

and letting $Y(s) \triangleq \mathfrak{L}[\Gamma y(x)]$ and $F(s) \triangleq \mathfrak{L}[\Gamma f(x)]$, the $\mathfrak{L}$ transformation gives

$$a_0 Y(s) + a_1 e^s[Y(s) - y(0)P(s)] \\ + a_2 e^{2s}[Y(s) - y(0)P(s) - y(1)e^{-s}P(s)] = F(s). \qquad [24]$$

Collecting terms and solving for $Y(s)$,

$$Y(s) = \frac{F(s) + y(0)(a_1 + a_2e^s)e^sP(s) + y(1)a_2e^sP(s)}{a_2e^{2s} + a_1e^s + a_0}. \qquad [25]$$

The $\mathfrak{L}^{-1}$ transformation of equation 25 gives the indicated solution

$$\text{Л}y(x) \;(=)\; \mathfrak{L}^{-1}[Y(s)], \qquad 0 \leqq x.$$

(*b*) Starting with the equation in the *difference* form,

$$b_2\Delta^2\text{Л}y(x) + b_1\Delta\text{Л}y(x) + b_0\text{Л}y(x) = \text{Л}f(x), \qquad [8]$$

the $\mathfrak{L}$ transformation gives

$$b_2[(e^s - 1)^2Y(s) - y(0)(e^s - 1)e^sP(s) - \Delta y(0)e^sP(s)]$$
$$+ b_1[(e^s - 1)Y(s) - y(0)e^sP(s)] + b_0Y(s) = F(s). \qquad [26]$$

Collecting terms and solving for $Y(s)$,

$$Y(s) = \frac{F(s) + y(0)[b_1 + b_2(e^s - 1)]e^sP(s) + \Delta y(0)b_2e^sP(s)}{b_2(e^s - 1)^2 + b_1(e^s - 1) + b_0}. \qquad [27]$$

Note that equation 27 reduces to equation 25 if the relations between the b's and a's given in equations 9 are used, and $\Delta y(0)$ is replaced by its equivalent, $y(1) - y(0)$, from 23. As with equation 25, the $\mathfrak{L}^{-1}$ transformation of 27 gives the indicated solution

$$\text{Л}y(x) \;(=)\; \mathfrak{L}^{-1}[Y(s)], \qquad 0 \leqq x.$$

Of the two procedures, that given in (*a*) is the easier to use, but that given in (*b*) discloses more clearly the analogy between the solution of difference equations and the solution of differential equations. Of course, with either procedure the solution obtained satisfies not only the given difference equation but also the prescribed boundary conditions. There are no arbitrary constants or undetermined periodic functions in the solution as with the classical method of solving difference equations. For a kth-order equation procedure (*a*) uses as boundary conditions the values that the unknown function is to have at k equally spaced points in the range $0 \leqq x \leqq k - 1$. Procedure (*b*) uses the values that the unknown function and its first $k - 1$ differences are to have in the interval $0 \leqq x \leqq 1$.

In effecting the solutions indicated in (*a*) and (*b*) it is seen that it will be necessary to find the $\mathfrak{L}$ transform of the driving function $\text{Л}f(x)$ and the $\mathfrak{L}^{-1}$ transform of the response function $Y(s)$. Just as it was convenient with differential equations to obtain the $\mathfrak{L}$ transforms of a number of elementary but key functions (Chapter 4) and later use these to obtain $\mathfrak{L}^{-1}$ transforms, so it will be convenient with difference

equations to build up a table of elementary-function transforms. Attention is turned to this in the following sections.

B. TRANSFORMS OF CERTAIN JUMP FUNCTIONS

In deriving the transforms of jump functions the approach will be made through the first difference,

$$\Delta \rfloor y(x) \triangleq \rfloor y(x+1) - \rfloor y(x), \qquad [4]$$

and its transform given in equation 19,

$$\mathfrak{L}[\Delta \rfloor y(x)] = (e^s - 1)\mathfrak{L}[\rfloor y(x)] - y(0)e^s P(s).$$

Although $\rfloor f(x)$ has been indicated as the driving function in the preceding difference equations, e.g., in 3 and 8, because the driving functions usually are jump functions, still it is possible to have driving functions that are not jump functions. Simplification will result if the transforms of such functions are expressed in the same form as that used for the transforms of jump functions. Consequently a preliminary example will show the transformation of a constant.

Example 1. Find $\mathfrak{L}[c]$, with c a real number. Here $f(x) \triangleq c$, and $\Delta c = c - c = 0$. Then

$$\mathfrak{L}[\Delta c] = 0,$$

$$(e^s - 1)\mathfrak{L}[c] - ce^s P(s) = 0,$$

therefore

$$\mathfrak{L}[c] = c\frac{e^s P(s)}{e^s - 1}, \qquad 0 < \sigma. \qquad [28]$$

7. POWER FUNCTIONS $\rfloor x^n$

Let $\rfloor y(x) \triangleq \rfloor x$ the graph of which is shown in Fig. 9·1. Then $\Delta \rfloor x = \rfloor (x+1) - \rfloor x = 1$, and

$$\mathfrak{L}[\Delta \rfloor x] = \mathfrak{L}[1],$$

$$(e^s - 1)\mathfrak{L}[\rfloor x] - 0 = \frac{e^s P(s)}{e^s - 1},$$

in which use is made of equation 28 with $c = 1$. Finally,

$$\mathfrak{L}[\rfloor x] = \frac{e^s P(s)}{(e^s - 1)^2}, \qquad 0 < \sigma. \qquad [29]$$

If $\rfloor y(x) \triangleq \rfloor x^2$, then $\Delta \rfloor x^2 = \rfloor (x+1)^2 - \rfloor x^2 = \rfloor 2x + 1$, and

$$\mathfrak{L}[\Delta \rfloor x^2] = \mathfrak{L}[\rfloor 2x + 1] = 2\mathfrak{L}[\rfloor x] + \mathfrak{L}[1],$$

$$(e^s - 1)\mathfrak{L}[\rfloor x^2] = 2\frac{e^s P(s)}{(e^s - 1)^2} + \frac{e^s P(s)}{e^s - 1}.$$

Therefore

$$\mathfrak{L}[\int x^2] = \frac{e^s P(s)}{(e^s - 1)^3}\,(e^s + 1), \qquad 0 < \sigma. \qquad [30]$$

By repetition of this process,

$$\mathfrak{L}[\int x^3] = \frac{e^s P(s)}{(e^s - 1)^4}\,(e^{2s} + 4e^s + 1), \qquad 0 < \sigma, \qquad [31]$$

$$\mathfrak{L}[\int x^4] = \frac{e^s P(s)}{(e^s - 1)^5}(e^{3s} + 11e^{2s} + 11e^s + 1), \qquad 0 < \sigma. \qquad [32]$$

8. Factorial Functions $\int x^{[n]}/n!$

The factorial

$$\int \frac{x^{[n]}}{n!} \triangleq \int \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!} \qquad [33]$$

has a simple transform which may be derived by induction, starting from the case $n = 2$.

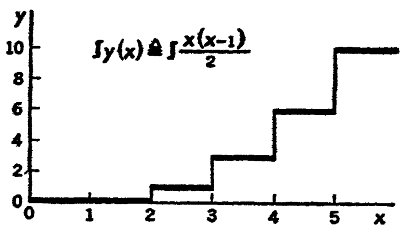


FIG. 9·2

For $n = 2$, $\int y(x) \triangleq \int \dfrac{x(x-1)}{2}$, and it has the form shown in Fig. 9·2. Then $\Delta \int \dfrac{x(x-1)}{2} = \int \dfrac{(x+1)x}{2} - \int \dfrac{x(x-1)}{2} = \int x$, and

$$\mathfrak{L}\left[\Delta \int \frac{x(x-1)}{2}\right] = \mathfrak{L}[\int x],$$

$$(e^s - 1)\mathfrak{L}\left[\int \frac{x(x-1)}{2}\right] - 0 = \mathfrak{L}[\int x].$$

Therefore

$$\mathfrak{L}\left[\int \frac{x^{[2]}}{2!}\right] = \frac{1}{e^s - 1}\,\mathfrak{L}[\int x] = \frac{e^s P(s)}{(e^s - 1)^3}, \qquad 0 < \sigma. \qquad [34]$$

For $n = 3$, $\int y(x) \triangleq \int \frac{x(x-1)(x-2)}{3!}$, then

$$\Delta \int \frac{x(x-1)(x-2)}{3!} = \int \frac{(x+1)x(x-1)}{3!} - \int \frac{x(x-1)(x-2)}{3!}$$

$$= \int \frac{x(x-1)}{2!},$$

and the $\mathcal{L}$ transformation gives

$$(e^s - 1)\,\mathcal{L}\left[\int \frac{x(x-1)(x-2)}{3!}\right] = \mathcal{L}\left[\int \frac{x(x-1)}{2!}\right].$$

Consequently

$$\mathcal{L}\left[\int \frac{x^{[3]}}{3!}\right] = \frac{1}{e^s - 1}\mathcal{L}\left[\int \frac{x^{[2]}}{2!}\right] = \frac{e^s P(s)}{(e^s - 1)^4}, \qquad 0 < \sigma. \qquad [35]$$

By iteration of this process,

$$\mathcal{L}\left[\int \frac{x^{[n]}}{n!}\right] = \frac{1}{e^s - 1}\mathcal{L}\left[\int \frac{x^{[n-1]}}{(n-1)!}\right] = \frac{e^s P(s)}{(e^s - 1)^{n+1}}, \qquad 0 < \sigma, \qquad [36]$$

in which n is a non-negative integer.

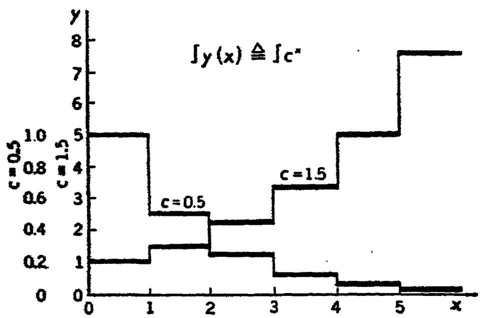


FIG. 9·3

9. EXPONENTIAL FUNCTIONS $\int c^x$

Let $\int y(x) \triangleq \int c^x$, in which c is a real number. If $c = 0$, $\int 0^x$ is defined here to be the unit pulse $p(x)$. As Heaviside said, "there are zeros and zeros." The form of $\int c^x$ is shown in Fig. 9·3. Then

$$\Delta \int c^x = \int c^{x+1} - \int c^x = (c-1)\int c^x.$$

Note that with jump functions of this type the difference has the same form as the original function. There is an analogy here to the deriva-

tive of the continuous exponential function. Proceeding to the transformation,

$$\mathfrak{L}[\Delta \int c^x] = (c - 1)\mathfrak{L}[\int c^x],$$
$$(e^s - 1)\mathfrak{L}[\int c^x] - e^s P(s) = (c - 1)\mathfrak{L}[\int c^x].$$

Combining terms,

$$(e^s - c)\mathfrak{L}[\int c^x] = e^s P(s).$$

Therefore
$$\mathfrak{L}[\int c^x] = \frac{e^s P(s)}{e^s - c}, \qquad \ln c < \sigma. \qquad [37]$$

With $c = e^a$, the base-e exponential jump functions are covered by equation 37. Thus

$$\mathfrak{L}[\int e^{ax}] = \frac{e^s P(s)}{e^s - e^a}, \qquad a < \sigma. \qquad [38]$$

10. Factorial-Exponential Products $\int \frac{x^{[n]} c^{x-n}}{n!}$

Let $\int y(x) \triangleq \int x c^x$; then

$$\Delta \int x c^x = \int (x + 1) c^{x+1} - \int x c^x = (c - 1) \int x c^x + c \int c^x,$$
$$\mathfrak{L}[\Delta \int x c^x] = (c - 1)\mathfrak{L}[\int x c^x] + c\mathfrak{L}[\int c^x],$$
$$(e^s - 1)\mathfrak{L}[\int x c^x] - 0 = (c - 1)\mathfrak{L}[\int x c^x] + c\frac{e^s P(s)}{e^s - c}.$$

Combining terms and solving for $\mathfrak{L}[\int x c^x]$,

$$\mathfrak{L}[\int x c^x] = c\frac{e^s P(s)}{(e^s - c)^2}, \qquad \ln c < \sigma; \qquad [39]$$

or collecting the c's,

$$\mathfrak{L}[\int x c^{x-1}] = \frac{e^s P(s)}{(e^s - c)^2}, \qquad \ln c < \sigma. \qquad [40]$$

The function $\int x c^{x-1}$ is c^{-1} times the product of functions of the type shown in Figs. 9·1 and 9·3.

If $\int y(x) \triangleq \int x^2 c^x$, then $\Delta \int x^2 c^x = (c - 1)\int x^2 c^x + 2c \int x c^x + c \int c^x$, and its $\mathfrak{L}$ transform is

$$(e^s - 1)\mathfrak{L}[\int x^2 c^x] = (c - 1)\mathfrak{L}[\int x^2 c^x] + 2c^2 \frac{e^s P(s)}{(e^s - c)^2} + c\frac{e^s P(s)}{e^s - c}.$$

Combining terms and solving for $\mathfrak{L}[\int x^2 c^x]$,

$$\mathfrak{L}[\int x^2 c^x] = 2c^2 \frac{e^s P(s)}{(e^s - c)^3} + c\frac{e^s P(s)}{(e^s - c)^2}, \qquad \ln c < \sigma; \qquad [41]$$

or substituting from equation 39 for the last term and rearranging,

$$\mathfrak{L}\left[\int \frac{x(x-1)c^{x-2}}{2!}\right] = \mathfrak{L}\left[\frac{x^{[2]}c^{x-2}}{2!}\right] = \frac{e^s P(s)}{(e^s - c)^3}, \qquad \ln c < \sigma. \qquad [42]$$

A graph of $\int x(x-1)c^{x-2}/2!$ is shown in Fig. 9·4.

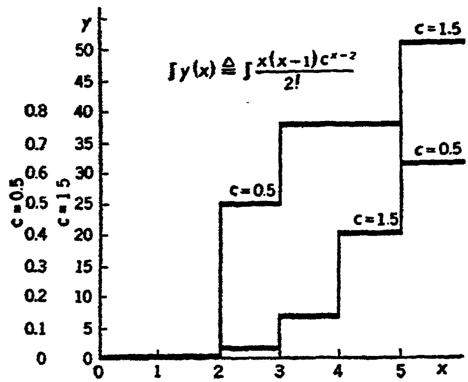


FIG. 9·4

By iteration of this process,

$$\mathfrak{L}\left[\int \frac{x^{[n]}c^{x-n}}{n!}\right] = \frac{e^s P(s)}{(e^s - c)^{n+1}}, \qquad \ln c < \sigma, \qquad [43]$$

in which n is a non-negative integer.

11. Sinusoidal Functions $\int \sin \beta x$

The transforms of sinusoidal jump functions can be found from the pair of difference equations,

$$\begin{aligned} \Delta \int \sin \beta x &= \int \sin \beta(x+1) - \int \sin \beta x \\ &= (\cos \beta - 1)\int \sin \beta x + \sin \beta \int \cos \beta x, \\ \Delta \int \cos \beta x &= \int \cos \beta(x+1) - \int \cos \beta x \\ &= (\cos \beta - 1)\int \cos \beta x - \sin \beta \int \sin \beta x. \end{aligned}$$

Transformation of these equations gives

$$\begin{aligned} (e^s - \cos \beta)\mathfrak{L}[\int \sin \beta x] &= \sin \beta\, \mathfrak{L}[\int \cos \beta x], \\ (e^s - \cos \beta)\mathfrak{L}[\int \cos \beta x] &= e^s P(s) - \sin \beta\, \mathfrak{L}[\int \sin \beta x]. \end{aligned}$$

Solving this set of equations,

$$\mathfrak{L}[\int \sin \beta x] = \frac{(\sin \beta) e^s P(s)}{e^{2s} - 2(\cos \beta) e^s + 1}, \qquad 0 < \sigma, \qquad [44]$$

$$\mathfrak{L}[\int \cos \beta x] = \frac{(e^s - \cos \beta) e^s P(s)}{e^{2s} - 2(\cos \beta) e^s + 1}, \qquad 0 < \sigma. \qquad [45]$$

The function $\int \sin \beta x$ is shown in Fig. 9·5.

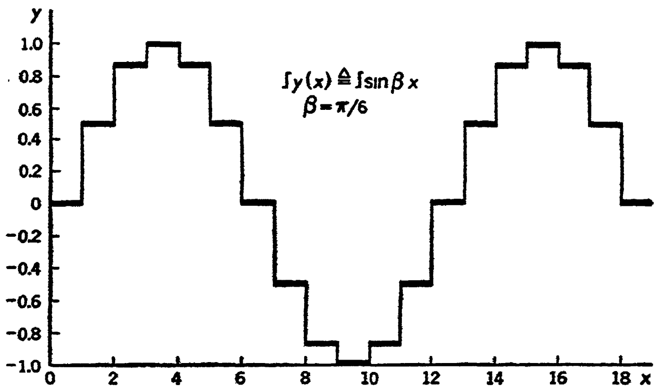


FIG. 9·5

Note that if $\beta = 2\pi$ the period is unity, and equations 44 and 45 degenerate to

$$\mathfrak{L}[\int \sin 2\pi x] = \mathfrak{L}[0] = 0,$$

$$\mathfrak{L}[\int \cos 2\pi x] = \mathfrak{L}[1] = \frac{e^s P(s)}{e^s - 1}.$$

C. INVERSE TRANSFORMATION OF FUNCTIONS OF e^s

Review of the transforms of jump functions which have been developed in the previous sections shows that they are all functions of e^s. This is in fact a characteristic of the transforms of all jump functions and for algebraic treatment of these transforms e^s rather than s may conveniently be taken as the variable. If the inverse transforms of functions of this type are not directly recognizable, they usually become so when the functions are expanded in a sum of simpler functions. Two types of expansions are helpful to this end: (1) expansions in partial fractions with e^s as the variable, and (2) expansions in series of powers of e^s.

12. EXPANSION IN PARTIAL FRACTIONS WITH e^s AS THE VARIABLE

In view of the discussion of partial-fraction expansions in Chapter 6, it will be sufficient here if the use of this method in the $\mathfrak{L}^{-1}$ transformation of functions of e^s is illustrated by two examples. In the first, the zeros of the denominator are different; in the second, one zero is repeated.

Example 1. Find the inverse transform of the function $\dfrac{e^sP(s)}{(e^s - c)(e^s - d)}$, in which c and d are real numbers.

Letting $p \triangleq e^s$ and making a partial-fraction expansion,

$$\frac{1}{(p - c)(p - d)} = \frac{1}{c - d}\left(\frac{1}{p - c} - \frac{1}{p - d}\right).$$

Based on this,

$$\frac{e^sP(s)}{(e^s - c)(e^s - d)} = \frac{e^sP(s)}{c - d}\left(\frac{1}{e^s - c} - \frac{1}{e^s - d}\right).$$

Then

$$\mathfrak{L}^{-1}\left[\frac{e^sP(s)}{(e^s - c)(e^s - d)}\right] = \frac{1}{c - d}\,\mathfrak{L}^{-1}\left[\frac{e^sP(s)}{e^s - c} - \frac{e^sP(s)}{e^s - d}\right]$$

$$(=)\ \frac{1}{c - d}\,\mathcal{J}(c^x - d^x), \qquad 0 \leqq x, \qquad [46]$$

using the pair given in equation 37. A plot of this jump function is shown in Fig. 9·6. Since both c and d have been chosen less than 1 it is analogous to

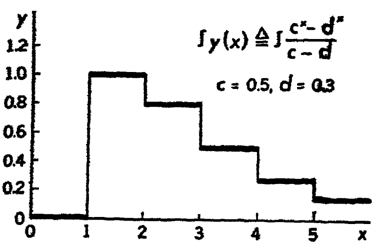


FIG. 9·6

the surge composed of the difference of two continuous exponentials having negative exponents.

Example 2. Find the inverse transform of the function $\dfrac{e^sP(s)}{(e^s - 1)^2(e^s - c)}$, in which c is a real number.

A partial-fraction expansion will be made, the form of the expansion being

determined by that of

$$\frac{1}{(\rho-1)^2(\rho-c)} = \frac{K_{11}}{(\rho-1)^2} + \frac{K_{12}}{\rho-1} + \frac{K_2}{\rho-c},$$

in which

$$K_{11} \triangleq \left[\frac{1}{\rho-c}\right]_{\rho=1} = \frac{1}{1-c},$$

$$K_{12} \triangleq \left[\frac{d}{d\rho}\frac{1}{\rho-c}\right]_{\rho=1} = \left[\frac{-1}{(\rho-c)^2}\right]_{\rho=1} = \frac{-1}{(1-c)^2},$$

$$K_2 \triangleq \left[\frac{1}{(\rho-1)^2}\right]_{\rho=c} = \frac{1}{(c-1)^2}.$$

Thus

$$\mathfrak{L}^{-1}\left[\frac{e^sP(s)}{(e^s-1)^2(e^s-c)}\right] = \mathfrak{L}^{-1}\left[\frac{1}{1-c}\frac{e^sP(s)}{(e^s-1)^2} - \frac{1}{(1-c)^2}\frac{e^sP(s)}{e^s-1} + \frac{1}{(c-1)^2}\frac{e^sP(s)}{e^s-c}\right]$$

$$(=) \int \frac{x}{1-c} - \frac{1}{(1-c)^2} + \int \frac{c^x}{(c-1)^2}$$

$$= \int \frac{c^x-1}{(c-1)^2} - \int \frac{x}{c-1}, \qquad 0 \leqq x, \qquad [47]$$

using pairs given in equations 29, 28, and 37. A plot of this jump function is shown in Fig. 9·7.

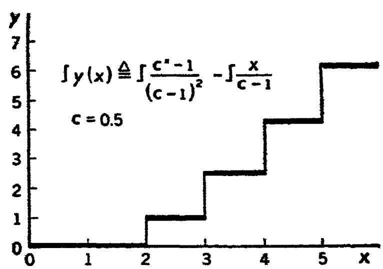


FIG. 9·7

13. EXPANSION IN SERIES OF POWERS OF e^s

The use of a series expansion of a transform as an aid to determining its inverse transform raises questions of the type cited in Sec. 5, Chapter 8. In particular, a change in the order of carrying out two limit processes must be justified. The method and reasoning can best be

made evident by carrying out the inverse transformation of a particular function, e.g., the function $\dfrac{e^s P(s)}{(e^s - c)(e^s - 1)}$, in which c is a real number.

Since

$$e^s - 1 \quad = 1 + e^{-s} + e^{-2s} + \cdots = \sum_{r=0} e^{-rs}, \tag{48}$$

then

$$\begin{aligned} \frac{e^s P(s)}{(e^s - c)(e^s - 1)} &= \frac{P(s)}{e^s - c} \sum_{r=0}^{\infty} e^{-} \\ &= P(s)(e^{-s} + ce^{-2s} + c^2 e^{-3s} + \cdots) \sum_{r=0} e^{-rs}. \end{aligned} \tag{49}$$

This may be written

$$\begin{aligned} P(s)(e^{-s} + ce^{-2s} + c^2 e^{-3s} &+ \cdots \\ + e^{-2s} + ce^{-3s} &+ \cdots \\ + e^{-3s} &+ \cdots \\ &+ \cdots), \end{aligned}$$

in which the first line corresponds to $r = 0$, the second line to $r = 1$, the third line to $r = 2$, etc. Collecting terms by columns, the result is

$$P(s)[e^{-s} + (1 + c)e^{-2s} + (1 + c + c^2)e^{-3s} + \cdots]. \tag{50}$$

The next step is to find the inverse transform of the infinite series in equation 50. Assuming that the order of summation and inverse transformation can be changed, the result of this change is

$$p(x - 1) + (1 + c)p(x - 2) + (1 + c + c^2)p(x - 3) + \cdots. \tag{51}$$

This jump function will be designated $\int y(x)$. The coefficient of $p(x - r)$ is

$$1 + c + c^2 + \cdots + c^{r-1} = \sum_{j=0}^{r-1} c^j = \frac{c^r - 1}{c - 1}, \qquad r = 1, 2, 3, \cdots, \tag{52}$$

since this is a finite geometric series. Consequently, as in equation 2,

$$\int y(x) \; (=) \; \sum_{r=1}^{\infty} \frac{c^r - 1}{c - 1} p(x - r) = \int \frac{c^x - 1}{c - 1}, \qquad 0 \leqq x. \tag{53}$$

Finally it is necessary to justify the formal interchange of limit processes made above. This will be done here by showing rigorously that

the direct transform of the result is the original function. Thus

$$\int_0^\infty \mathcal{J}\frac{c^x-1}{c-1}e^{-sx}dx = \frac{1}{c-1}\left[\int_0^\infty \mathcal{J}c^x e^{-sx}dx - \int_0^\infty e^{-sx}dx\right]$$

$$= \frac{1}{c-1}\left[\frac{e^sP(s)}{e^s-c} - \frac{e^sP(s)}{e^s-1}\right] = \frac{e^sP(s)}{(e^s-c)(e^s-1)},$$

$$\max(0, \ln c) < \sigma. \qquad [54]$$

Use has been made of pairs given in equations 37 and 28. Since an agreement is obtained, the result in equation 53 is correct, i.e.,

$$\mathfrak{L}^{-1}\left[\frac{e^sP(s)}{(e^s-c)(e^s-1)}\right](=)\,\mathcal{J}\frac{c^x-1}{c-1}, \qquad 0 \leq x. \qquad [55]$$

A plot of $\mathcal{J}(c^x-1)/(c-1)$ is shown in Fig. 9·8.

Although the series in equation 51 is a solution of the original problem, its sum in closed form given in equation 53 is more useful. Comment should be made, however, on the connecting step indicated in equation 52.

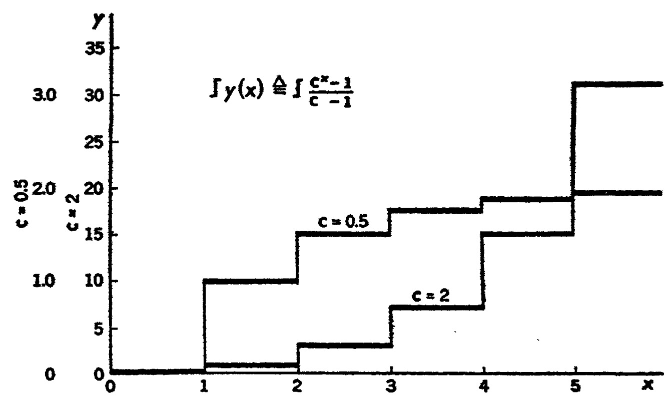


FIG. 9·8

The summation of a finite series to obtain a typical coefficient as in equation 52 can occasionally be done simply; more frequently it is a step of considerable difficulty. The problem of summing a finite series is equivalent to the problem of finding another function whose first difference is the typical summand. The analogous statement for differentiable functions would be: The problem of integrating a function is equivalent to the problem of finding another function whose first deriva-

tive is the integrand. Thus the problem of summing the finite series

$$\sum_{j=0}^{r-1} c^j \tag{56}$$

given in equation 52 is equivalent to the problem of finding a function $f(j)$ whose first difference,

$$f(j+1) - f(j) \triangleq \Delta f(j), \tag{57}$$

is equal to the typical summand c^j. In other words, it is necessary to solve the difference equation

$$\Delta f(j) = c^j. \tag{58}$$

Since the solution of difference equations is the general topic of this chapter, it would appear that the sum, if not readily recognized, might be carried out by solving the equivalent difference equation as a side problem. The substitution of a difference problem for a summation problem is entirely logical and analogous to the substitution of a derivative problem for an integral problem, but this substitution is not helpful in a problem such as the one arising in this example because the transform of $f(j)$ in equation 58 is identical with the function with which the example started if $f(0) = 0$. Here then, another method of obtaining the sum is necessary. Fortunately in this particular example a method was available which was not dependent on a partial-fraction expansion. The sum could be obtained by the usual procedure for summing a finite geometric series using its ratio and first and last terms.

14. Effect of Multiplying or Dividing by e^s or $(e^s - 1)$

The principles expressed in Theorem 10 should be kept in mind when treating the transforms of jump functions since they provide checks and shortcuts in $\mathfrak{L}^{-1}$ transformations.

There is a limit to the amount of translation of $\int y(x)$ to the *left* which can be effected by multiplication of $\mathfrak{L}[\int y(x)]$ by a power of e^s. After such a multiplication the degree of e^s in the numerator must not exceed the degree of e^s in the denominator. Expressed in another way, the function $\int y(x)$ before translation to the left must be zero to the right of the origin for an interval at least equal to the amount of the translation.

For example, with a a non-negative integer,

$$e^{as}\mathfrak{L}\left[\int \frac{x^{[3]}}{3!}\right] = e^{as}\frac{e^s P(s)}{(e^s-1)^4} = \mathfrak{L}\left[\int \frac{(x+a)^{[3]}}{3!}\right] \tag{59}$$

for $a \leqq 3$, but

$$e^{as}\mathfrak{L}[\jmath x^3] = e^{as}\frac{e^s P(s)}{(e^s - 1)^4}(e^{2s} + 4e^s + 1) = \mathfrak{L}[\jmath(x + a)^3] \qquad [60]$$

for $a = 1$ only.

Although in contrast with this there is no finite limit to the amount of translation of $\jmath y(x)$ to the *right* by multiplication of $\mathfrak{L}[\jmath y(x)]$ by a power of e^{-s}, the function after translation must be zero to the right of the origin for an interval equal to the amount of the translation.

For example, with a a non-negative integer,

$$e^{-as}\mathfrak{L}\left[\jmath\frac{x^{[3]}}{3!}\right] = \mathfrak{L}\left[\jmath\frac{(x - a)^{[3]}}{3!}u(x - a)\right], \qquad [61]$$

and

$$e^{-as}\mathfrak{L}[\jmath x^3] = \mathfrak{L}[\jmath(x - a)^3 u(x - a)]. \qquad [62]$$

The factor $u(x - a)$ is an essential part of the translated function.

The principles expressed in Theorems 6–a and 7–a have their analogies in the transform theory of jump functions. If $\jmath y(x)$ is a jump function and $y_d(x)$ is a differentiable function, multiplication of $\mathfrak{L}[\jmath y(x)]$ by $(e^s - 1)$ corresponds to "differencing" (the operation represented by Δ) *provided* $\jmath y(x)$ is zero for the unit interval beginning with $x = 0$, just as multiplication of $\mathfrak{L}[y_d(x)]$ by s corresponds to differentiation provided $y_d(x)$ is zero at $x = 0$. Thus for jump functions,

$$(e^s - 1)\mathfrak{L}[\jmath y(x)] = \mathfrak{L}[\Delta\jmath y(x)], \qquad \text{if } y(0) = 0. \qquad [63]$$

For this to hold, the degree of $(e^s - 1)$ in the numerator of $(e^s - 1)\mathfrak{L}[\jmath y(x)]$ must not exceed the degree of $(e^s - 1)$ in the denominator.

Division of $\mathfrak{L}[\jmath y(x)]$ by $(e^s - 1)$ corresponds to finite summation (the operation represented by $\sum\limits_{j=0}^{r}$) just as division of $\mathfrak{L}[y_d(x)]$ by s corresponds to integration from 0 to x. The result after summing is always zero for the unit interval beginning with $x = 0$, just as the result of integrating is zero at $x = 0$.

D. EXAMPLES

15. First-Order Equations

Example 1. Find the jump function which satisfies the difference equation

$$\jmath y(x + 1) - a\jmath y(x) = c, \qquad [64]$$

with $y(0) = \mu$; a, c, and μ are real numbers.

Assume that $\int y(x)$ is $\mathfrak{L}$ transformable with $\mathfrak{L}[\int y(x)] \triangleq Y(s)$, and transform equation 64.

$$\mathfrak{L}[\int y(x+1)] - a\mathfrak{L}[\int y(x)] = \mathfrak{L}[c].$$

Using the pairs given in equations 12 and 28,

$$e^s[Y(s) - y(0)P(s)] - aY(s) = c\,\frac{e^sP(s)}{e^s - 1},$$

or

$$(e^s - a)Y(s) = c\,\frac{e^sP(s)}{e^s-1} + \mu e^sP(s).$$

Solving for $Y(s)$,

$$Y(s) = c\,\frac{e^sP(s)}{(e^s-1)(e^s-a)} + \mu\,\frac{e^sP(s)}{e^s-a}.$$

Then

$$\mathfrak{L}^{-1}[Y(s)] = \mathfrak{L}^{-1}\left[c\,\frac{e^sP(s)}{(e^s-1)(e^s-a)} + \mu\,\frac{e^sP(s)}{e^s-a}\right].$$

$$\int y(x)\ (=)\ c\int\frac{a^x-1}{a-1} + \mu\int a^x, \qquad 0 \leqq x. \tag{65}$$

In carrying out the inverse transformation use has been made of pairs given in equations 55 and 37.

The result given in equation 65 is a solution of 64, since

$$\int y(x+1) - a\int y(x) = c\int\frac{aa^x-1}{a-1} + \mu a\int a^x - ac\int\frac{a^x-1}{a-1} - \mu a\int a^x = c.$$

Furthermore it satisfies the boundary condition, since $y(0) = \mu$.

Example 2. Solve the difference equation,

$$\Delta\int y(x) = \int x^2, \tag{66}$$

with $y(0) = \mu$; μ is a real constant.

Let $\mathfrak{L}[\int y(x)] \triangleq Y(s)$, and transform equation 66.

$$\mathfrak{L}[\Delta\int y(x)] = \mathfrak{L}[\int x^2].$$

With the aid of pairs given in equations 19 and 30, this can be written

$$(e^s - 1)Y(s) - y(0)e^sP(s) = \frac{e^sP(s)}{(e^s-1)^3}(e^s+1).$$

Then

$$Y(s) \quad \frac{e^sP(s)}{(e^s-1)^4}(e^s+1) + \mu\,\frac{e^sP(s)}{e^s-1}$$

$$= e^s\,\frac{e^sP(s)}{(e^s-1)^4} + \frac{e^sP(s)}{(e^s-1)^4} + \mu\,\frac{e^sP(s)}{e^s-1}$$

$$= e^s\mathfrak{L}\left[\int\frac{x^{[3]}}{3!}\right] + \mathfrak{L}\left[\int\frac{x^{[3]}}{3!}\right] + \mathfrak{L}[\mu], \tag{67}$$

using pairs given in equations 35 and 28. The $\mathfrak{L}^{-1}$ transformation gives

$$\int y(x) \;(=)\; \int \frac{(x+1)^{[3]}}{3!} + \int \frac{x^{[3]}}{3!} + \mu, \qquad 0 \leqq x. \tag{68}$$

In finding the first term of equation 68 use has been made of equation 59.

16. Second-Order Equation

Example 1. Solve the second-order difference equation

$$\int y(x+2) - 5\int y(x+1) + 6\int y(x) = \int x, \tag{69}$$

with $y(0) = \mu_0$ and $y(1) = \mu_1$.

Assume that $\int y(x)$ is $\mathfrak{L}$ transformable with $\mathfrak{L}[\int y(x)] \triangleq Y(s)$ and transform equation 69, applying pairs given in equations 17, 12, and 29,

$$\begin{aligned} &e^{2s}[Y(s) - y(0)P(s) - y(1)e^{-s}P(s)] \\ &\quad - 5e^{s}[Y(s) - y(0)P(s)] + 6Y(s) = \frac{e^{s}P(s)}{(e^{s}-1)^2}. \end{aligned}$$

Then transposing and collecting terms,

$$(e^{2s} - 5e^{s} + 6)Y(s) = \frac{e^{s}P(s)}{(e^{s}-1)^2} + \mu_0 e^{s}(e^{s}-5)P(s) + \mu_1 e^{s}P(s). \tag{70}$$

With $\rho \triangleq e^{s}$ the characteristic equation is

$$\rho^2 - 5\rho + 6 = 0,$$

the roots of which are 2 and 3. In terms of these roots,

$$Y(s) = \frac{e^{s}P(s)}{(e^{s}-2)(e^{s}-3)(e^{s}-1)^2} + \mu_0\frac{e^{s}(e^{s}-5)P(s)}{(e^{s}-2)(e^{s}-3)} + \mu_1\frac{e^{s}P(s)}{(e^{s}-2)(e^{s}-3)}$$

An expansion of $Y(s)$ will be made based on the forms

$$\frac{}{(\rho-2)(\rho-3)} = \frac{}{\rho-3} - \frac{}{\rho-2},$$

$$\frac{\rho-5}{(\rho-2)(\rho-3)} = \frac{3}{\rho-2} - \frac{2}{\rho-3}.$$

The expansion is

$$\begin{aligned} Y(s) = \left(\frac{1}{e^{s}-3} - \frac{1}{e^{s}-2}\right)\frac{e^{s}P(s)}{(e^{s}-1)^2} &+ \mu_0 e^{s}P(s)\left(\frac{3}{e^{s}-2} - \frac{2}{e^{s}-3}\right) \\ &+ \mu_1 e^{s}P(s)\left(\frac{1}{e^{s}-3} - \frac{1}{e^{s}-2}\right). \end{aligned}$$

The $\mathfrak{L}^{-1}$ transformation of $Y(s)$ gives, using pairs stated in equations 47 and 37,

$$\int y(x)\,(=)\,\int\left(\frac{3^x-1}{4}-\frac{x}{2}\right)-\int(2^x-1-x)+\mu_0\int(3\cdot 2^x-2\cdot 3^x)$$
$$+\,\mu_1\int(3^x-2^x)$$
$$=A\int 3^x+B\int 2^x+\frac{1}{2}\int x+\frac{3}{4},\qquad 0\leq x, \qquad [71]$$

in which $A \triangleq \frac{1}{4} - 2\mu_0 + \mu_1$ and $B \triangleq -1 + 3\mu_0 - \mu_1$.

17. Sawtooth-Voltage Generator

It has been pointed out in Sec. 14, Chapter 7, that in general the $\mathfrak{L}$-transformation method loses part of its advantage over the classical method of solving problems when the boundary conditions are divided between two points. In that section it was shown how a two-point boundary-value problem can be solved by the introduction of literal boundary values. In the present section another procedure for handling two-point boundary-value problems will be presented. It consists in introducing a jump function in an auxiliary variable. This jump function enters because time is divided into a succession of equal intervals and the boundary values at the successive points of division can be interpolated by a jump function. The jump function is found by solving a difference equation. This auxiliary difference equation can be solved by the $\mathfrak{L}$-transformation method presented in this chapter. This procedure for handling a two-point boundary-value problem necessitates $\mathfrak{L}$ transformations with respect to both the principle and the auxiliary independent variables. A simple example follows.

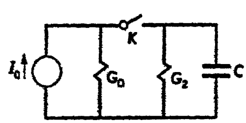


FIG. 9·9. The cyclic closing and opening of K produces a sawtooth-voltage drop across C.

An elementary generator of a sawtooth-voltage drop consists of a parallel circuit of capacitance C and conductance G_2 which is connected cyclically to a parallel combination of a constant-current source I_0 and a conductance G_0. (See Fig. 9·9.) For simplicity the electronic control that does this is represented by the switch K which in each cycle is first closed for t_1 seconds and is then held open for $t_2 - t_1$ seconds. The problem is to calculate (1) the build-up of voltage across C, beginning with C uncharged, and (2) the approximate number of cycles necessary to reach a steady state, i.e., a repeated-transient state.

In the nth cycle, let $v_1(t)$ be the condenser voltage while K is closed and let $v_2(t)$ be the condenser voltage while K is open. For simplicity, t will be measured from the beginning of the nth cycle.

The differential equations are

$$\left.\begin{aligned} C\frac{dv_1}{dt} + G_1v_1 &= I_0, \qquad 0 \leqq t \leqq t_1 \\ C\frac{dv_2}{dt} + G_2v_2 &= 0, \qquad t_1 \leqq t \leqq t_2 \end{aligned}\right\} \tag{72}$$

in which $G_1 \triangleq G_0 + G_2$. The value of v_1 at t_1 is the initial value of v_2 because the magnitude of the condenser voltage is continuous. Since these are simple differential equations only their solutions will be given. If the equations were more complicated it would be necessary to use the direct and inverse transformations to obtain the form of the solutions. The solutions of equations 72 are

$$\left.\begin{aligned} v_1(t) &= \frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t}) + \gamma(n)e^{-\alpha_1 t}, \qquad 0 \leqq t \leqq t_1 \\ v_2(t) &= v_1(t_1)e^{-\alpha_2(t-t_1)}, \qquad t_1 \leqq t \leqq t_2. \end{aligned}\right\} \tag{73}$$

Here $\alpha_1 \triangleq G_1/C$, $\alpha_2 \triangleq G_2/C$, and $\gamma(n)$ is the initial value of the condenser voltage for the nth cycle.

The final value of the condenser voltage for the nth cycle is $v_2(t_2)$. Since the magnitude of the condenser voltage is continuous, $v_2(t_2)$ is the initial value $\gamma(n + 1)$ of the condenser voltage for the $(n + 1)$st cycle. Thus

$$\begin{aligned} \gamma(n + 1) &= v_2(t_2) \\ &= \left[\frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t_1}) + \gamma(n)e^{-\alpha_1 t_1}\right]e^{-\alpha_2(t_2-t_1)}. \end{aligned}$$

This simplifies to

$$\gamma(n + 1) - a\gamma(n) = b, \tag{74}$$

in which

$$a \triangleq e^{-\alpha_1 t_1}e^{-\alpha_2(t_2-t_1)},$$

$$b \triangleq \frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t_1})e^{-\alpha_2(t_2-t_1)}.$$

The function $\gamma(n)$ has so far been defined only for integral values of n. Its definition can be extended to non-integral values of n by interpolating the original point function by a jump function which passes through these points. This jump function will be designated by $\mathfrak{J}\gamma(n)$. Equation 74 then becomes

$$\mathfrak{J}\gamma(n + 1) - a\mathfrak{J}\gamma(n) = b. \tag{75}$$

$\gamma(0)$ is zero since the condenser is uncharged at the start. The solution of equation 75 can be taken from Example 1, Sec. 15. It is

$$\lceil\!\rfloor\gamma(n)\ (=)\ b\lceil\!\rfloor\frac{a^n - 1}{a - 1}, \qquad 0 \leqq n. \tag{76}$$

Substitution of result 76 in the solutions of equations 73 gives the condenser voltage in the nth cycle. It should be recalled that t was chosen zero at the start of the nth cycle.

$$\left.\begin{aligned} v_1(t) &= \frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t}) + b\frac{a^n - 1}{a - 1}e^{-\alpha_1 t}, \qquad 0 \leqq t \leqq t_1 \\ v_2(t) &= \left[\frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t_1}) + b\frac{a^n - 1}{a - 1}e^{-\alpha_1 t_1}\right]e^{-\alpha_2(t - t_1)}, \\ &\qquad\qquad t_1 \leqq t \leqq t_2 \end{aligned}\right\} \tag{77}$$

Since n is a second independent variable, this expression gives the voltage for cycles during the transient build-up as well as for cycles during the steady state. During the transient build-up the voltage waveform may be described as a transient sawtooth, and during the steady state as a steady-state sawtooth.

Solution for the transient build-up without the use of a difference equation would have been a laborious step-by-step procedure. The introduction of a difference equation shortened the procedure and caused little difficulty since the $\mathfrak{L}$-transformation method solves difference equations by the same sequence of steps that are already familiar for the solution of i-d equations.

When n is large, $a^n \to 0$, and equations 77 become those for the steady state,

$$\left.\begin{aligned} v_1(t) &= \frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t}) - \frac{b}{a - 1}e^{-\alpha_1 t}, \qquad 0 \leqq t \leqq t_1 \\ v_2(t) &= \left[\frac{I_0}{C\alpha_1}(1 - e^{-\alpha_1 t_1}) - \frac{b}{a - 1}e^{-\alpha_1 t_1}\right]e^{-\alpha_2(t - t_1)}, \quad t_1 \leqq t \leqq t_2. \end{aligned}\right\} \tag{78}$$

Note that now $v_2(t_2) = v_1(0)$. The approximate number of cycles needed before the steady state is reached can be calculated from the requirement that n must be great enough to make $a^n \ll 1$.

18. Transform Equations of a General Ladder Network

In the problem of the previous section a difference equation arose because time was divided into successive equal intervals by switching. In the problem of the present section a system of difference equations

arises because the physical system consists of a chain of identical sections [CA 8, AL 1, MA 4, WH 1].

Consider a ladder-type network composed of a succession of identical T sections, one of which is represented diagrammatically in Fig. 9·10. This might be taken to represent, for example, a section of an artificial transmission line [CA 8, KU 2] or an electric filter [CA 7].

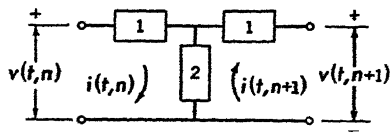


FIG. 9·10. The $n + 1$ st section of a ladder-type network composed of identical T sections.

The instantaneous terminal voltage drops and the loop currents of the section are indicated on the diagram. Note that there is a second independent variable — the ordinal number of the section terminal — which is indicated along with time in the function notation. Let

a_1 be the i-d operator for the 2-terminal series branches indicated by 1,

a_2 be the i-d operator for the 2-terminal shunt branch indicated by 2.

The i-d difference equations for the T section are

$$\left.\begin{aligned}(a_1 + a_2)i(t,n) - a_2 i(t,n+1) &= v(t,n),\\ -a_2 i(t,n) + (a_1 + a_2)i_2(t,n+1) &= -v(t,n+1).\end{aligned}\right\} \qquad [79]$$

Assume that the energy stored in the network is zero initially.

Since there are two independent variables there will be a need for two direct transformations, one $\mathfrak{L}_t$ with respect to t and the other $\mathfrak{L}_n$ with respect to n. It is immaterial which of these transformations is carried out first since t and n are independent variables. In the $\mathfrak{L}_t$ transformation the initial conditions are introduced into the equations; in the $\mathfrak{L}_n$ transformation the terminal conditions are introduced.

If $\mathfrak{L}_t[i(t,n)] \triangleq I(s,n)$ and $\mathfrak{L}_t[v(t,n)] \triangleq V(s,n)$, the $\mathfrak{L}_t$ transformation of equations 79 yields — since the initial conditions are all specified as zero —

$$\left.\begin{aligned}(z_1 + z_2)I(s,n) - z_2 I(s,n+1) &= V(s,n),\\ -z_2 I(s,n) + (z_1 + z_2)I(s,n+1) &= -V(s,n+1).\end{aligned}\right\} \qquad [80]$$

Here

z_1 is the impedance function for each of the 2-terminal series branches,

z_2 is the impedance function for the 2-terminal shunt branch.

Equations 80 are now difference equations in n and algebraic equations in s. To make these difference equations suceptible of $\mathfrak{L}$ transforma-

tion with respect to n the definition of $I(s,n)$ and $V(s,n)$ as functions of n will be extended so that n can take on all real non-negative values rather than just integral values. This will be done by interpolating the points of $I(s,n)$ and $V(s,n)$ by jump functions starting at $n = 0$. These new functions will be denoted by $\int I(s,n)$ and $\int V(s,n)$, respectively. Equations 80 then become

$$\left.\begin{aligned} (z_1 + z_2)\int I(s,n) - z_2\int I(s,n+1) &= \int V(s,n), \\ -z_2\int I(s,n) + (z_1 + z_2)\int I(s,n+1) &= -\int V(s,n+1). \end{aligned}\right\} \quad [81]$$

These equations will now be transformed with respect to n.

Let $\mathfrak{L}_n[\int I(s,n)] \triangleq \mathcal{I}(s,w)$ and $\mathfrak{L}_n[\int V(s,n)] \triangleq \mathcal{V}(s,w)$, the script capital letters now indicating a transform with respect to n. The complex variable w will correspond to n, just as s corresponds to t. The $\mathfrak{L}_n$ transformation of equations 81 gives

$$\left.\begin{aligned} &(z_1 + z_2)\mathcal{I}(s,w) - z_2e^w[\mathcal{I}(s,w) - I(s,0)P(w)] = \mathcal{V}(s,w), \\ &-z_2\mathcal{I}(s,w) + (z_1 + z_2)e^w[\mathcal{I}(s,w) - I(s,0)P(w)] \\ &\qquad\qquad\qquad = -e^w[\mathcal{V}(s,w) - V(s,0)P(w)]. \end{aligned}\right\} \quad [82]$$

The transform equations 82 incorporate all the information contained originally in the i-d difference equations and the statement of initial conditions; they provide for the specification of terminal conditions. They can be solved algebraically for $\mathcal{V}(s,w)$ and $\mathcal{I}(s,w)$, which are next $\mathfrak{L}^{-1}$ transformed with respect to w, and then $\mathfrak{L}^{-1}$ transformed with respect to s. This does more than simply "unwind" what has previously been done — it yields the solution to the boundary-value problem specified.

Collecting terms in equations 82,

$$\left.\begin{aligned} &(z_1 + z_2 - z_2e^w)\mathcal{I}(s,w) - \mathcal{V}(s,w) = -I(s,0)z_2e^wP(w), \\ &[(z_1 + z_2)e^w - z_2]\mathcal{I}(s,w) + e^w\mathcal{V}(s,w) = I(s,0)(z_1 + z_2)e^wP(w) \\ &\qquad\qquad\qquad + V(s,0)e^wP(w). \end{aligned}\right\} \quad [83]$$

Solving equations 83,

$$\left.\begin{aligned} \mathcal{V}(s,w) &= \frac{V(s,0)(e^w - \lambda) - I(s,0)z_2(\lambda^2 - 1)}{e^{2w} - 2\lambda e^w + 1}\, e^wP(w), \\ \mathcal{I}(s,w) &= \frac{I(s,0)(e^w - \lambda) - V(s,0)z_2^{-1}}{e^{2w} - 2\lambda e^w + 1}\, e^wP(w), \end{aligned}\right\} \quad [84]$$

in which $\lambda \triangleq (z_1 + z_2)/z_2$.

With $\rho \triangleq e^w$, the characteristic equation is

$$\rho^2 - 2\lambda\rho + 1 = 0.$$

Its roots are $\rho_1 \triangleq \lambda + \sqrt{\lambda^2 - 1}$ and $\rho_2 \triangleq \lambda - \sqrt{\lambda^2 - 1} = 1/\rho_1$. The fractions in equations 84 can be expanded in partial fractions having the form

$$\frac{\rho - \lambda}{(\rho - \rho_1)(\rho - \rho_2)} = \frac{1}{\rho_1 - \rho_2}\left(\frac{\rho_1 - \lambda}{\rho - \rho_1} - \frac{\rho_2 - \lambda}{\rho - \rho_2}\right) = \frac{1}{2}\left(\frac{1}{\rho - \rho_1} + \frac{1}{\rho - \rho_2}\right),$$

$$\frac{1}{(\rho - \rho_1)(\rho - \rho_2)} = \frac{1}{\rho_1 - \rho_2}\left(\frac{1}{\rho - \rho_1} - \frac{1}{\rho - \rho_2}\right)$$

$$= \frac{1}{2\sqrt{\lambda^2 - 1}}\left(\frac{1}{\rho - \rho_1} - \frac{1}{\rho - \rho_2}\right).$$

Based on these expansions, equations 84 become

$$\left.\begin{aligned}\mathcal{V}(s,w) &= \frac{V(s,0)}{2}\left(\frac{1}{e^w - \rho_1} + \frac{1}{e^w - \rho_2}\right)e^w P(w)\\ &\quad - \frac{I(s,0)z_2\sqrt{\lambda^2 - 1}}{2}\left(\frac{1}{e^w - \rho_1} - \frac{1}{e^w - \rho_2}\right)e^w P(w),\\ \mathcal{G}(s,w) &= \frac{I(s,0)}{2}\left(\frac{1}{e^w - \rho_1} + \frac{1}{e^w - \rho_2}\right)e^w P(w)\\ &\quad - \frac{V(s,0)}{2z_2\sqrt{\lambda^2 - 1}}\left(\frac{1}{e^w - \rho_1} - \frac{1}{e^w - \rho_2}\right)e^w P(w).\end{aligned}\right\} \quad [85]$$

Since, by the pair given in equation 37,

$$\mathfrak{L}_w^{-1}\left[\frac{e^w P(w)}{e^w - \rho_k}\right] (=) \mathfrak{I}\rho_k^n,$$

the $\mathfrak{L}_w^{-1}$ transformation of equations 85 yields for $0 \leqq n$,

$$\left.\begin{aligned}\mathfrak{I}V(s,n) &(=) V(s,0)\mathfrak{I}\frac{\rho_1^n + \rho_2^n}{2} - ZI(s,0)\mathfrak{I}\frac{\rho_1^n - \rho_2^n}{2},\\ \mathfrak{I}I(s,n) &(=) I(s,0)\mathfrak{I}\frac{\rho_1^n + \rho_2^n}{2} - YV(s,0)\mathfrak{I}\frac{\rho_1^n - \rho_2^n}{2},\end{aligned}\right\} \quad [86]$$

in which $Z \triangleq z_2\sqrt{\lambda^2 - 1} = \sqrt{(z_1 + z_2)^2 - z_2^2}$ and $Y \triangleq Z^{-1}$. Z is the *iterative impedance function* and Y is the *iterative admittance function.*

Recalling that $\rho \triangleq e^{w}$, ρ_1 would be written e^{w_1} and since $\rho_2 = 1/\rho_1$ it is seen that ρ_2 would be e^{-w_1}. If these values were substituted in equations 86 the combinations of ρ_1 and ρ_2 would be hyperbolic cosine and sine of nw_1. Replacing w_1 by the more conventional symbol β it will be observed that $\cosh \beta = \lambda$ and $\sinh \beta = \sqrt{\lambda^2 - 1}$.

Upon substitution of hyperbolic functions in equations 86 and considering only integral values of n, the jump notation may be dropped and these equations become

$$V(s,n) = V(s,0) \cosh n\beta - ZI(s,0) \sinh n\beta, \tag{87}$$

$$I(s,n) = I(s,0) \cosh n\beta - YV(s,0) \sinh n\beta. \tag{88}$$

Equations 87 and 88 are the general transform equations for the ladder-type network of T sections in which the initial distribution of current and voltage is zero. If desired, these equations may be taken as the starting point for the solution of any particular terminal-condition problem on a network of the ladder type with T sections having zero initial energy storage. It may be added in passing that if the initial energy storage had not been zero, its only effect would have been to add two terms each to equations 87 and 88 and to add complication to their subsequent solution.

Finally the $\mathfrak{L}_s^{-1}$ transformation of equations 87 and 88 gives for $0 \leq t$,

$$\left.\begin{aligned} v(t,n) &\ (=)\ \mathfrak{L}_s^{-1}[V(s,n)], \\ i(t,n) &\ (=)\ \mathfrak{L}_s^{-1}[I(s,n)]. \end{aligned}\right\} \tag{89}$$

This is an indicated solution of the system of two i-d difference equations 79, assuming zero initial conditions. Until z_1, z_2, and two terminal conditions are specified the solution cannot be evaluated explicitly.

19. Transform Equations for Short-Circuited Ladder Network

As a simple example to show how the terminal conditions are inserted in the general transform equations of the ladder network, assume that the network is short-circuited at the Nth termination.

For this boundary condition $v(t,N) = 0$, consequently $V(s,N) = 0$, and equation 87 becomes

$$0 = V(s,0) \cosh N\beta - ZI(s,0) \sinh N\beta,$$

from which

$$I(s,0) = \frac{V(s,0)}{Z} \frac{\cosh N\beta}{\sinh N\beta}. \tag{90}$$

Substitution from equation 90 in equations 87 and 88 gives

$$V(s,n) = V(s,0)\cosh n\beta - V(s,0)\frac{\cosh N\beta \cdot \sinh n\beta}{\sinh N\beta}$$

$$= V(s,0)\frac{\sinh (N-n)\beta}{\sinh N\beta}. \qquad [91]$$

$$I(s,n) = \frac{V(s,0)}{Z}\frac{\cosh N\beta \cdot \cosh n\beta}{\sinh N\beta} - YV(s,0)\sinh n\beta$$

$$= YV(s,0)\frac{\cosh (N-n)\beta}{\sinh N\beta}. \qquad [92]$$

Equations 91 and 92 are now in form for $\mathfrak{L}_s^{-1}$ transformation. Rather than carry this further they will be used to show how the current and voltage are established in a long ladder network when the network is subjected suddenly to a constant voltage at terminal $n = 0$, the initial currents and voltages of the network being zero. For this it will be desirable to work with a "semi-infinite" line, i.e., one extending from the origin to infinity in one direction. Such a line is a mathematical abstraction but it is useful because it eliminates the complication of reflections from the far end.

20. Response of Semi-Infinite Ladder Network to Unit Step Voltage

Writing equation 91 in exponential form and letting $N \to \infty$,

$$V(s,n) = \lim_{N\to\infty} V(s,0)\frac{e^{(N-n)\beta} - e^{-(N-n)\beta}}{e^{N\beta} - e^{-N\beta}}$$

$$= \lim_{N\to\infty} V(s,0)\frac{e^{-n\beta} - e^{-(2N-n)\beta}}{1 - e^{-2N\beta}}$$

$$= V(s,0)e^{-n\beta}, \qquad \text{if } 0 < \mathfrak{R}[\beta]. \qquad [93]$$

Similarly from equation 92,

$$I(s,n) = YV(s,0)e^{-n\beta}. \qquad [94]$$

Equations 93 and 94 are the transform equations for voltage and current, respectively, in a semi-infinite ladder network. The results would have been the same if an open-circuited line had been used.

Since, from equations 93 and 94,

$$\frac{V(s,n+1)}{V(s,n)} = e^{-\beta} \quad \text{and} \quad \frac{I(s,n+1)}{I(s,n)} = e^{-\beta},$$

it can be seen that the new variable β that was introduced in equations 87 and 88 is the function of s that governs the progress of the disturbance along the network. β is called the *propagation function.*

It is necessary now, if the analysis is to be carried further, to specify the structure of the network and the form of the applied voltage or current at terminal $n = 0$.

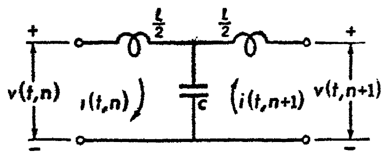


FIG. 9·11. The $n + 1$ st section of an artificial transmission line, neglecting losses.

Example 1. Assume that each T section of an artificial transmission line, neglecting losses, is composed as shown in Fig. 9·11 and that a unit step voltage is applied at terminal $n = 0$.

Here $v(t,0) = u(t)$, $z_1 = Ls/2$, $z_2 = 1/Cs$;

$$\lambda \triangleq \frac{z_1 + z_2}{z_2} = \frac{LC}{2}\left(s^2 + \frac{2}{LC}\right) = \frac{2}{\gamma^2}\left(s^2 + \frac{\gamma^2}{2}\right), \quad \text{with } \gamma \triangleq \frac{2}{\sqrt{LC}};$$

$$Z \triangleq z_2\sqrt{\lambda^2 - 1} = \frac{1}{Cs}\sqrt{\frac{4}{\gamma^4}\left(s^2 + \frac{\gamma^2}{2}\right)^2 - 1} = \frac{L}{2}\sqrt{s^2 + \gamma^2};$$

$$\rho_1 \triangleq \lambda + \sqrt{\lambda^2 - 1} = \frac{2}{\gamma^2}\left[\left(s^2 + \frac{\gamma^2}{2}\right) + s\sqrt{s^2 + \gamma^2}\right] = \frac{1}{\gamma^2}(\sqrt{s^2 + \gamma^2} + s)^2;$$

$$e^{-n\rho} = \rho_2^n = \frac{1}{\rho_1^n} = \frac{1}{(\lambda + \sqrt{\lambda^2 - 1})^n} - - \quad \gamma^{2n}$$

Then $V(s,0) = 1/s$, and from equations 93 and 94,

$$V(s,n) = \frac{\gamma^{2n}}{s(\sqrt{s^2 + \gamma^2} + s)^{2n}}, \tag{95}$$

$$I(s,n) = \frac{2\gamma^{2n}}{Ls\sqrt{s^2 + \gamma^2}\,(\sqrt{s^2 + \gamma^2} + s)^{2n}}. \tag{96}$$

The $\mathfrak{L}^{-1}$ transformation of equations 95 and 96 presents a problem which has not heretofore been discussed — the inverse transformation of irrational functions. Although each section of the network is composed only of lumped constants, the ladderlike connection of an infinite number of these sections gives rise to an infinite system of similar *i-d* equations which are summarized by two *i-d* difference equations 79. These in turn give rise to current and voltage transforms that are irrational functions. The $\mathfrak{L}^{-1}$ transforms of these particular irrational functions involve Bessel functions and will be derived in the following section. The results found there will be used here to complete

the present example. Thus the instantaneous voltage and current at any section are

$$v(t,n) \;(=)\; \mathfrak{L}^{-1}\left[\frac{\gamma^{2n}}{s(\sqrt{s^2+\gamma^2}+s)^{2n}}\right]$$

$$(=)\; 2n\int_0^t \frac{J_{2n}(\gamma t)}{t}\,dt, \qquad 0 \leqq t, \qquad n \neq 0, \qquad [97]$$

$$i(t,n) \;(=)\; \mathfrak{L}^{-1}\left[\frac{2\gamma^{2n}}{Ls\sqrt{s^2+\gamma^2}\,(\sqrt{s^2+\gamma^2}+s)^{2n}}\right]$$

$$(=)\; \frac{2}{L}\int_0^t J_{2n}(\gamma t)dt, \qquad 0 \leqq t. \qquad [98]$$

For a discussion of transients in ladder-type networks the reader is referred to [CA 6, 7, 8, AL 1, BA 1, BO 7].

21. TRANSFORMS OF INTEGRAL-ORDER BESSEL FUNCTIONS

The irrational functions for $\mathfrak{L}[v(t,n)]$ and $\mathfrak{L}[i(t,n)]$ of a semi-infinite ladder network which were derived in the previous section are too complicated to investigate directly. It will be necessary first to find the $\mathfrak{L}^{-1}$ transform of a simple irrational function of s. The result will be used to form a key pair from which other more complicated pairs, including those of immediate concern, will be derived.

The $\mathfrak{L}^{-1}$ transform of the simple irrational function $(s^2+\alpha^2)^{-\frac{1}{2}}$, α real, will be found by first expanding the function in a series of descending powers of s, using the binomial expansion.

$$\mathfrak{L}^{-1}\left[\frac{1}{\sqrt{s^2+\alpha^2}}\right] = \mathfrak{L}^{-1}\left[\frac{1}{s}\left(1+\frac{\alpha^2}{s^2}\right)^{-\frac{1}{2}}\right]$$

$$= \mathfrak{L}^{-1}\left[\frac{1}{s}\left(1-\frac{\alpha^2}{2s^2}+\frac{3\alpha^4}{2^2 2!s^4}-\frac{3.5\alpha^6}{2^3 3!s^6}+\cdots\right)\right]$$

$$= \mathfrak{L}^{-1}\left[\frac{1}{s}-\frac{\alpha^2}{2s^3}+\frac{3\alpha^4}{2^2 2!s^5}-\frac{3.5\alpha^6}{2^3 3!s^7}+\cdots\right]. \qquad [99]$$

Now assuming that the order of summation and $\mathfrak{L}^{-1}$ transformation can be changed, there is obtained the series

$$\mathfrak{L}^{-1}\left[\frac{1}{\sqrt{s^2+\alpha^2}}\right] (=)\; 1-\frac{\alpha^2t^2}{2.2!}+\frac{3\alpha^4t^4}{2^2 2!4!}-\frac{3.5\alpha^6t^6}{2^3 3!6!}+\cdots$$

$$= 1-\frac{(\alpha t)^2}{2^2}+\frac{(\alpha t)^4}{2^2 4^2}-\frac{(\alpha t)^6}{2^2 4^2 6^2}+\cdots. \qquad [100]$$

This can be identified as the series for the Bessel function [GR 1, MC 1, WA 8] of the first kind and zero order, which is denoted by $J_0(\alpha t)$. That $J_0(\alpha t)$ is the inverse transform of $(s^2 + \alpha^2)^{-\frac{1}{2}}$ can be verified by showing that the direct transform of $J_0(\alpha t)$ is $(s^2 + \alpha^2)^{-\frac{1}{2}}$. The desired relation,

$$\int_0^\infty J_0(\alpha t)e^{-st}dt = \frac{1}{\sqrt{s^2 + \alpha^2}}, \qquad 0 < \sigma, \qquad [101]$$

is known [WA 8].

The series expansion is an effective method of finding the $\mathfrak{L}^{-1}$ transform of an s-function when other simpler methods fail. If the resulting series in t can be identified as one whose properties are known and whose values have been tabulated or charted, the series expansion method is especially useful. In case the series cannot be identified but can be demonstrated to possess useful convergent or asymptotic properties, it can be used for numerical computation, although this is often a tedious procedure. Fortunately, series treatments are usually not needed for finite lumped-constant systems. Expansion in a series might be necessary to effect the transformation of the driving function. It also might be necessary in carrying out the inverse transformation that provides the forced solution.

Having obtained the key pair,

$$\mathfrak{L}[J_0(\alpha t)] = \frac{1}{\sqrt{s^2 + \alpha^2}}, \qquad [102]$$

additional pairs can be developed by application to it of (1) the theorems for $\mathfrak{L}$-transformable functions and (2) the recursion formulas which relate the simple zero-order to higher-order Bessel functions.

Applying Theorem 6 (real differentiation) to equation 102 and using the formula giving the derivative of a zero-order Bessel function, and using $J_0(0) = 1$,

$$\mathfrak{L}\left[\frac{d}{dt} J_0(\alpha t)\right] = s\mathfrak{L}[J_0(\alpha t)] - J_0(0),$$

$$\mathfrak{L}[-\alpha J_1(\alpha t)] = \frac{s}{\sqrt{s^2 + \alpha^2}} - 1 = \frac{s - \sqrt{s^2 + \alpha^2}}{\sqrt{s^2 + \alpha^2}}$$

Rationalizing the numerator, this yields the transform of the first-order Bessel function $J_1(\alpha t)$,

$$\mathfrak{L}[J_1(\alpha t)] = \frac{\alpha}{\sqrt{s^2 + \alpha^2}(\sqrt{s^2 + \alpha^2} + s)}, \qquad 0 < \sigma. \qquad [103]$$

With n a non-negative integer, the nth-order Bessel function $J_n(\alpha t)$ satisfies the difference-differential equation,

$$J_{n+1}(\alpha t) - J_{n-1}(\alpha t) = -2\frac{dJ_n(\alpha t)}{d(\alpha t)},$$

the conventional subscript notation being used here in place of the function notation, i.e., $J_n(\alpha t) \equiv J(\alpha t, n)$. Although this difference-differential equation can be solved for $\mathfrak{L}_t[J_n(\alpha t)]$ by the transformation method presented above, it is simpler here to use the equation as a recursion formula. This equation, after the order is stepped down by unity, is

$$J_n(\alpha t) = J_{n-2}(\alpha t) - 2\frac{dJ_{n-1}(\alpha t)}{d(\alpha t)}. \qquad [104]$$

The $\mathfrak{L}$ transformation of equation 104, with $n = 2$, gives

$$\mathfrak{L}[J_2(\alpha t)] = \mathfrak{L}[J_0(\alpha t)] - \frac{2}{\alpha}\mathfrak{L}\left[\frac{dJ_1(\alpha t)}{dt}\right].$$

Applying the real differentiation theorem to the last term and using the fact that $J_1(0) = 0$,

$$\begin{aligned}\mathfrak{L}[J_2(\alpha t)] &= \frac{1}{\sqrt{s^2+\alpha^2}} - \frac{2s}{\alpha}\mathfrak{L}[J_1(\alpha t)] + 0 \\ &= \frac{1}{\sqrt{s^2+\alpha^2}} - \frac{2s}{\sqrt{s^2+\alpha^2}(\sqrt{s^2+\alpha^2}+s)} \\ &= \frac{\alpha^2}{\sqrt{s^2+\alpha^2}(\sqrt{s^2+\alpha^2}+s)^2}, \qquad 0 < \sigma. \qquad [105]\end{aligned}$$

By an iteration of this process, there is obtained

$$\mathfrak{L}[J_n(\alpha t)] = \frac{\alpha^n}{\sqrt{s^2+\alpha^2}(\sqrt{s^2+\alpha^2}+s)^n}, \qquad \begin{matrix} 0 < \sigma, \\ n = 0, 1, 2, 3, \cdots. \end{matrix} \qquad [106]$$

To obtain the pair that was used in equation 98 for the build-up of current in a ladder network, apply Theorem 7–a (division by s) to equation 106.

$$\mathfrak{L}\left[\int_0^t J_n(\alpha t)dt\right] = \frac{\alpha^n}{s\sqrt{s^2+\alpha^2}(\sqrt{s^2+\alpha^2}+s)^n}, \qquad \begin{matrix} 0 < \sigma, \\ n = 0, 1, 2, 3, \cdots. \end{matrix} \qquad [107]$$

A second difference equation for Bessel functions of integral order is

$$\frac{J_n(\alpha t)}{\alpha t} = \frac{1}{2n}[J_{n-1}(\alpha t) + J_{n+1}(\alpha t)]. \qquad [108]$$

As with equation 104, this will be used as a recursion formula. The $\mathfrak{L}$ transformation of equation 108, with $n = 1$, gives

$$\mathfrak{L}\left[\frac{J_1(\alpha t)}{t}\right] = \frac{\alpha}{2}\{\mathfrak{L}[J_0(\alpha t)] + \mathfrak{L}[J_2(\alpha t)]\}$$

$$2\left[\frac{1}{\sqrt{s^2+\alpha^2}} + \frac{\alpha^2}{\sqrt{s^2+\alpha^2}(\sqrt{s^2+\alpha^2}+s)^2}\right]$$

$$-\frac{\alpha}{\sqrt{s^2+\alpha^2}+s}, \qquad 0 < \sigma. \qquad [109]$$

By an iteration of this process there is derived

$$\mathfrak{L}\left[\frac{nJ_n(\alpha t)}{t}\right] = \frac{\alpha^n}{(\sqrt{s^2+\alpha^2}+s)^n}, \qquad \begin{matrix} 0 < \sigma, \\ n = 1, 2, 3, \cdots. \end{matrix} \qquad [110]$$

The pair that was applied in equation 97 for the build-up of voltage in a ladder network is obtained from equation 110 by application of Theorem 7–a (division by s).

$$\mathfrak{L}\left[\int_0^t \frac{nJ_n(\alpha t)}{t}\,dt\right] = \frac{\alpha^n}{s(\sqrt{s^2+\alpha^2}+s)^n}, \qquad \begin{matrix} 0 < \sigma, \\ n = 1, 2, 3, \cdots. \end{matrix} \qquad [111]$$

Although several other transform pairs for Bessel functions might be obtained by proceeding in this way, their development will be deferred until they are needed in Volume 2.

22. Transient Voltages in Ladder-Network Representation of a Transformer Winding

As a final example of the use of difference equations, the transient voltage distribution in a ladder-network representation of a transformer winding will be developed. The example is chosen to illustrate several troublesome points that may arise in using a partial-fraction expansion as an aid to inverse transformation.

The winding of a transformer or of a machine is actually a distributed-constant system, but if an approximate analysis of the winding's behavior when subjected suddenly to changes of terminal voltage is to be made it is convenient to use in place of the distributed-constant system an approximate equivalent network having lumped constants and a finite

number of loops. When the number of loops taken is large, however, analysis by ordinary procedure becomes too cumbersome to carry out. Even with the method shown in Sec. 9, Chapter 6, for obtaining the roots of the characteristic equation from the system determinant, the labor is still great and practically prohibitive when the number of loops is ten or more. This problem yields easily, however, if attacked by use of difference equations. With its aid the magnitude of each space harmonic, and such functions as the space distribution at any instant, and the time variation at any point of the voltage to ground can be computed readily.

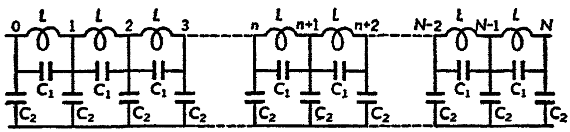


FIG. 9·12. Ladder-network representation of a transformer winding.

In Fig. 9·12, L represents the inductance of a coil, C_1 represents the capacitance between adjacent coils, and C_2 represents the capacitance of a coil to ground. All coils are assumed to be alike.

Since the ladder network of Sec. 18 was analyzed on the loop basis, this network will be analyzed on the node basis. The dependent variables will be the node voltages, a typical one of which is $v(t,n)$.

Let $\mathfrak{L}_t[v(t,n)] \triangleq V(s,n)$, $n = 0, 1, 2, \cdots, N$. The self-admittance function for node n is $y_0 \triangleq [(2C_1 + C_2)s + 2/Ls]$; the mutual-admittance function for two adjacent nodes is $y_1 \triangleq (C_1 s + 1/Ls)$.

The transform equation for the $(n + 1)$st node is

$$-y_1 V(s,n+2) + y_0 V(s,n+1) - y_1 V(s,n) = 0. \qquad [112]$$

This simplifies to

$$V(s,n+2) - 2\lambda V(s,n+1) + V(s,n) = 0, \qquad [113]$$

in which

$$\lambda \triangleq \frac{y_0}{2y_1} = \frac{\sigma^2}{\gamma^2} \cdot \frac{s^2 + \gamma^2}{s^2 + \sigma^2},$$

with $\gamma^2 \triangleq 2/L(2C_1 + C_2)$ and $\sigma^2 \triangleq 1/LC_1$. This is a second-order difference equation and is the result of the $\mathfrak{L}_t$ transformation of the i-d difference equation for a typical node, the initial conditions all being zero.

To make the difference equation 113 susceptible of $\mathfrak{L}$ transformation with respect to n the definition of $V(s,n)$ as a function of n will be extended so that n can take on all real non-negative values rather than

just integral values. This will be done by interpolating the points of $V(s,n)$ by a jump function starting at $n = 0$. This new function will be denoted by $\mathcal{J}V(s,n)$. Equation 113 then becomes

$$\mathcal{J}V(s,n+2) - 2\lambda\mathcal{J}V(s,n+1) + \mathcal{J}V(s,n) = 0. \qquad [114]$$

This equation will now be transformed with respect to n.

Let $\mathfrak{L}_n[\mathcal{J}V(s,n)] \triangleq \mathcal{V}(s,w)$; then the $\mathfrak{L}_n$ transformation of equation 114 gives

$$e^{2w}[\mathcal{V}(s,w) - V(s,0)P(w) - V(s,1)e^{-w}P(w)] \\ -2\lambda e^{w}[\mathcal{V}(s,w) - V(s,0)P(w)] + \mathcal{V}(s,w) = 0. \qquad [115]$$

Solving for $\mathcal{V}(s,w)$,

$$\mathcal{V}(s,w) = \frac{V(s,1) + V(s,0)(e^w - 2\lambda)}{e^{2w} - 2\lambda e^w + 1}\, e^w P(w). \qquad [116]$$

Letting $\rho \triangleq e^w$, the characteristic equation is

$$\rho^2 - 2\lambda\rho + 1 = 0.$$

Its roots are $\rho_1 \triangleq e^{w_1} \triangleq \lambda + \sqrt{\lambda^2 - 1}$ and $\rho_2 \triangleq \lambda - \sqrt{\lambda^2 - 1} = e^{-w_1}$. Replacing w_1 by the more conventional symbol β it will be observed that $\lambda = \cosh\beta$ and $\sqrt{\lambda^2 - 1} = \sinh\beta$.

The form for the partial-fraction expansion of equation 116 depends upon

$$\frac{1}{(\rho - \rho_1)(\rho - \rho_2)} = \frac{1}{2\sqrt{\lambda^2 - 1}}\left(\frac{1}{\rho - \rho_1} - \frac{1}{\rho - \rho_2}\right),$$

$$\frac{\rho - 2\lambda}{(\rho - \rho_1)(\rho - \rho_2)} = -\frac{1}{2\sqrt{\lambda^2 - 1}}\left(\frac{\rho_2}{\rho - \rho_1} - \frac{\rho_1}{\rho - \rho_2}\right).$$

Thus equation 116 can be written

$$\mathcal{V}(s,w) = \frac{V(s,1)}{2\sqrt{\lambda^2 - 1}}\left(\frac{1}{e^w - \rho_1} - \frac{1}{e^w - \rho_2}\right) e^w P(w) \\ \frac{V(s,0)}{2\sqrt{\lambda^2 - 1}}\left(\frac{\rho_2}{e^w - \rho_1} - \frac{\rho_1}{e^w - \rho_2}\right) e^w P(w). \qquad [117]$$

Since, for ρ_k a constant,

$$\mathfrak{L}_w^{-1}\left[\frac{e^w P(w)}{e^w - \rho_k}\right] (=) \mathcal{J}\rho_k^n,$$

the $\mathfrak{L}_w^{-1}$ transformation of equation 117 yields

$$\mathcal{J}V(s,n) (=) \frac{V(s,1)}{\sqrt{\lambda^2 - 1}}\mathcal{J}\frac{\rho_1^n - \rho_2^n}{2} - \frac{V(s,0)}{\sqrt{\lambda^2 - 1}}\mathcal{J}\frac{\rho_1^{n-1} - \rho_2^{n-1}}{2}, \\ 0 \le n. \qquad [118]$$

Recalling that $\rho_1 = e^{\beta}$ and $\rho_2 = e^{-\beta}$, in which β has replaced w_1, the combinations of ρ_1 and ρ_2 in equation 118 can be written as $\sinh n\beta$ and $\sinh (n-1)\beta$. Upon substitution of these expressions in equation 118 and considering only integral values of n, the jump notation may be dropped, and the equation becomes

$$V(s,n) = \frac{V(s,1)\sinh n\beta - V(s,0)\sinh (n-1)\beta}{\sinh \beta}. \qquad [119]$$

To represent a winding grounded at one end, let the network be grounded at the Nth node. Then $v(t,N) = 0$, making $V(s,N) = 0$. Using this boundary condition in equation 119,

$$0 = V(s,1)\sinh N\beta - V(s,0)\sinh (N-1)\beta,$$

or

$$V(s,1) = V(s,0)\frac{\sinh (N-1)\beta}{\sinh N\beta}. \qquad [120]$$

Substituting from equation 120 in 119,

$$V(s,n) = V(s,0)\frac{\sinh (N-1)\beta\cdot\sinh n\beta - \sinh N\beta\cdot\sinh (n-1)\beta}{\sinh\beta\cdot\sinh N\beta}$$

$$= V(s,0)\frac{\sinh (N-n)\beta}{\sinh N\beta}. \qquad [121]$$

If the voltage applied to node 0 is a unit step function, $v(t,0) = u(t)$ and $V(s,0) = 1/s$. Then equation 121 becomes

$$V(s,n) = \frac{\sinh (N-n)\beta}{s\sinh N\beta} \triangleq \frac{A(s)}{sB_1(s)}. \qquad [122]$$

Preparatory to finding the inverse transform of $V(s,n)$ this function will be expanded in partial fractions. The characteristic equation,

$$\sinh N\beta = 0,$$

is satisfied by $\beta_k = \pm jk\pi/N$, $k = 0, 1, 2, \cdots$. The values of s corresponding to these values of β_k can be found from the equation for β,

$$\lambda - \cosh\beta = 0$$

$$\frac{\sigma^2}{\gamma^2}\cdot\frac{s^2+\gamma^2}{s^2+\sigma^2} - \cosh\frac{\pm jk\pi}{N} = 0$$

$$\left(\frac{\sigma^2}{\gamma^2} - \cos\frac{k\pi}{N}\right)s^2 + \sigma^2\left(1 - \cos\frac{k\pi}{N}\right) = 0. \qquad [123]$$

The values of s are

$$s_{k_1},s_{k_2} \triangleq \pm j\sigma \left[\frac{1-\cos\dfrac{k\pi}{N}}{\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}}\right] \triangleq \pm j\omega_k, \tag{124}$$

with $k = 0, 1, 2, \cdots, N$. Several points need comment: (1) The $\pm$ signs in the argument for the cosine were dropped because the cosine is an even function. (2) The upper limit for k is N because no new roots, beyond those already found, result from k's greater than N. (3) Since $s_0 = 0$ is also a zero of $A(s)$ in equation 122, the fraction $A(s)/sB_1(s)$ has only a first-order pole at the origin.

The partial-fraction expansion of equation 122 can be written

$$\begin{aligned} V(s,n) &= \frac{A(0)}{B_1(0)} + \sum_{k=1}^{N}\frac{K_k}{s-j\omega_k} + \sum_{k=1}^{N}\frac{\bar{K}_k}{s+j\omega_k} \\ &= \frac{A(0)}{B_1(0)} + \sum_{k=1}^{N}\frac{(K_k+\bar{K}_k)s + j\omega_k(K_k-\bar{K}_k)}{s^2+\omega_k^2}, \end{aligned} \tag{125}$$

in which $0 \leqq n \leqq N$, and

$$\frac{A(0)}{B_1(0)} = \frac{N-n}{N} = 1-\frac{n}{N},$$

$$\begin{aligned} K_k &\triangleq \left[\frac{(s-j\omega_k)A(s)}{sB_1(s)}\right]_{s=j\omega_k} = \left[\frac{A(s)}{sB_1'(s)}\right]_{s=j\omega_k} \\ &= \left[\frac{\sinh (N-n)\beta}{Ns\dfrac{d\beta}{ds}\cosh N\beta}\right]. \end{aligned}$$

Since $\beta = \cosh^{-1}\lambda$ and $\lambda \triangleq \dfrac{\sigma^2}{\gamma^2}\cdot\dfrac{s^2+\gamma^2}{s^2+\sigma^2}$,

$$\begin{aligned} \frac{d\beta}{ds} &= \frac{d\beta}{d\lambda}\cdot\frac{d\lambda}{ds} \\ &= \frac{1}{\sqrt{\lambda^2-1}}\cdot\frac{\sigma^2}{\gamma^2}\cdot\frac{(s^2+\sigma^2)2s-2s(s^2+\gamma^2)}{(s^2+\sigma^2)^2} \end{aligned}$$

$$\frac{2s}{\quad}\cdot\frac{\sigma^2}{\gamma^2}-\lambda \qquad = \frac{2s}{s^2+\sigma^2}\cdot\frac{\dfrac{\sigma^2}{\gamma^2}-\cosh\beta}{\sinh\beta}$$

If s has the value s_k, then β has the value β_k. Consequently

$$\left[s\frac{d\beta}{ds}\right]_{s=j\omega_k} = \frac{-2\omega_k^2}{-\omega_k^2+\sigma^2}\cdot\frac{\dfrac{\sigma^2}{\gamma^2}-\cosh\dfrac{\pm jk\pi}{N}}{\sinh\dfrac{\pm jk\pi}{N}}$$

$$= \frac{2\omega_k^2}{\omega_k^2-\sigma^2}\cdot\frac{\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}}{j\sin\dfrac{\pm k\pi}{N}}$$

$$= \frac{2\left(1-\cos\dfrac{k\pi}{N}\right)}{1-\dfrac{\sigma^2}{\gamma^2}}\cdot\frac{\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}}{j\sin\dfrac{\pm k\pi}{N}}.$$

In addition

$$\left[\frac{\sinh (N-n)\beta}{\cosh N\beta}\right]_{s=j\omega_k} = \frac{\sinh N\beta_k\cdot\cosh n\beta_k - \cosh N\beta_k\cdot\sinh n\beta_k}{\cosh N\beta_k}$$

$$= -\sinh\frac{\pm jnk\pi}{N} = -j\sin\frac{\pm nk\pi}{N}.$$

Then a typical coefficient is

$$K_k \triangleq \frac{\left(1-\dfrac{\sigma^2}{\gamma^2}\right)\left(j\sin\dfrac{\pm k\pi}{N}\right)\left(-j\sin\dfrac{\pm nk\pi}{N}\right)}{2N\left(1-\cos\dfrac{k\pi}{N}\right)\left(\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}\right)}$$

$$= \frac{\left(1-\dfrac{\sigma^2}{\gamma^2}\right)\sin\dfrac{k\pi}{N}\cdot\sin\dfrac{nk\pi}{N}}{2N\left(1-\cos\dfrac{k\pi}{N}\right)\left(\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}\right)}.$$

The $\pm$ signs are dropped because the sine is an odd function, and the product of two odd functions is an even function.

Since K_k is a real number, $\overline{K}_k = K_k$ and equation 125 reduces to

$$V(s,n) = \left(1-\frac{n}{N}\right) + \sum_{k=1}^{N}\frac{2K_k s}{s^2+\omega_k^2} \qquad [126]$$

Finally, the $\mathfrak{L}_s^{-1}$ transformation of equation 126 gives for $0 \leqq t$ and $0 \leqq n \leqq N$,

$$v(t,n)\ (=)\left(1-\frac{n}{N}\right)+\sum_{k=1}^{N}2K_k\cos\omega_k t, \qquad [127]$$

in which

$$2\,K_k \triangleq \frac{1-\dfrac{\sigma^2}{\gamma^2}}{N}\cdot\frac{\sin\dfrac{k\pi}{N}\cdot\sin\dfrac{nk\pi}{N}}{\left(1-\cos\dfrac{k\pi}{N}\right)\left(\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}\right)}$$

No question about changing the order of inverse transformation and summation arises because the number of terms in the sum is finite. The characteristic angular frequencies as found in equation 124 are

$$\omega_k \triangleq \sigma\sqrt{\frac{1-\cos\dfrac{k\pi}{N}}{\dfrac{\sigma^2}{\gamma^2}-\cos\dfrac{k\pi}{N}}} \qquad k=0,1,2,\cdots,N.$$

In equations 125 to 127 the term for $k = 0$ is the constant term $1 - n/N$.

Equation 127 gives the voltage to ground at any node along the ladder-network representation (Fig. 9·12) of a winding having one end grounded and the other subjected to a unit step voltage, the winding being initially without currents or charges. Although the solution is good for any finite number of nodes N, its advantage over other lumped-constant solutions for this ladder-type network is greatest if N is large.

PROBLEMS

9·1. If a personal loan of \$300 is repaid by making 20 monthly payments of \$19.65, what is the uniform rate of interest paid on the unpaid principal? Answer this by first solving the difference equation for the problem.

9·2. On the first of each month (30 days) a payment of c dollars is made into an annuity fund whose interest is compounded daily at a decimal rate r. Express this in a difference equation. From the solution of this equation find the principal in the fund at the end of 6 months if on January 1 the principal including the January 1 payment is μ dollars.

9·3. (*a*) A mortgage is retired by making uniform monthly payments consisting of interest and payments on the unpaid principal. If the original principal is μ, the monthly decimal interest rate is r, and the payment made the first of each month is c, find the amount of the unpaid principal after the first of any month.

(*b*) If it is a 20-year mortgage and the interest rate is 5 percent, what must be the uniform monthly payment per thousand of original principal?

9·4. Over a period of 20 years a certain manufacturing company receives from sales an income per year that varies in accordance with the function $A(c^x - d^x)$ in which A is a constant, c and d are constants less than 1, and x is the number of the year. If on each January 1 it adds to its surplus fund the fraction q of its income from sales the previous year, and the surplus fund draws interest at the decimal rate r, how much is there in the surplus fund at the end of 20 years? The initial amount in the fund is μ.

9·5. A manufacturing company plans to place on the market a new device with a 3-month guarantee of satisfactory service. When a defective unit is returned a replacement unit is to be shipped promptly, all replacement units to be guaranteed likewise for 3 months. The company estimates that of the total number of units shipped in any month 6 percent will be returned for replacement during the succeeding month, 3 percent during the second succeeding month, and 1 percent during the third succeeding month. If it maintains a selling schedule of a units per month, what will be the ratio of units sold to total units shipped during any month?

9·6. If, in Prob. 9·5, the selling schedule should grow in accordance with the relation $a(1 - 0.9^x)$ units per month, in which a is a constant and x is the number of months from the time the device is first introduced, how must the production per month be built up to supply the shipments?

9·7. Show that:

$$(a)\quad \mathfrak{L}[\int c^x \sin \beta x] = \frac{(c \sin \beta)e^s P(s)}{(e^s - c \cos \beta)^2 + (c \sin \beta)^2}.$$

$$(b)\quad \mathfrak{L}[\int c^x \cos \beta x] = \frac{(e^s - c \cos \beta)e^s P(s)}{(e^s - c \cos \beta)^2 + (c \sin \beta)^2}.$$

9·8. Solve the following difference equations:

$$(a)\quad \int y(x+1) - 2\int y(x) = 5\int \sin \frac{\pi}{6} x$$

with $y(0) = 3$.

$$(b)\quad \Delta \int y(x) = 10 \int x(0.8)^x$$

with $y(0) = -2$.

$$(c)\quad \int y(x+2) - 3\int y(x+1) + 2\int y(x) = 10\int 0.5^x$$

with $y(0) = -4$ and $y(1) = -2$.

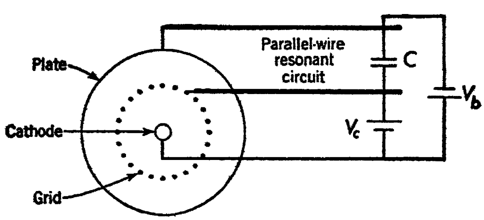


FIG. 9·P9

9·9. In the figure is shown a positive-grid oscillator for producing (inefficiently) oscillations of 1000 megacycles per second. At this high frequency the transit time of the electrons in the interelectrode space is a controlling factor. The external circuit which is tuned to resonate with the electron oscillations is a parallel-wire transmission line.

Consider a group of electrons emitted by the cathode during a brief interval. This group is accelerated toward the grid because of the positive grid potential. Some of the electrons hit the grid and are lost. The remainder, because of their momentum, pass through the grid and proceed toward the plate but are slowed down by the retarding field. Of these, a few strike the plate and are lost; the remainder come to rest, reverse their direction, and are accelerated back toward the positive grid. Some again are lost by hitting the grid. Those that pass through are slowed down as they approach the cathode. This cycle may be repeated many times before all the electrons in the group are lost to the grid or the plate.

While this has been taking place other groups have been supplied by the cathode. The resonant circuit permits a circulation of induced currents, and the voltage drop between grid and plate serves to synchronize the oscillations of the electrons. Those electrons emitted in unfavorable phase relation with respect to this synchronizing voltage do not contribute to the oscillations. As a result the useful electrons supplied by the cathode can be assumed to come off in pulses, a electrons in each pulse, and one pulse each cycle.

In the build-up of the oscillations how does the number of electrons in the space between grid and plate vary from cycle to cycle? It may be assumed that in each passage of a group of electrons through the grid 50 percent are captured by the grid. The number lost to the plate may be neglected. The number present in the grid-plate space before the oscillations are synchronized is b.

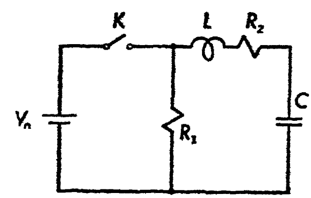


FIG. 9·P10

9·10. In the network shown in the diagram, K is a switch which is closed for t_1 seconds and then opened for $t_2 - t_1$ seconds, this sequence being repeated cyclically. When K closes in the nth cycle, $\gamma(n)$ is the initial voltage across C and $\rho(n)$ is the initial current in L. (a) Show that these initial values are related by a set of first-order difference equations of the form

$$\int\gamma(n+1) + a_1\int\gamma(n) + b_1\int\rho(n) = c_1$$
$$\int\rho(n+1) + a_2\int\rho(n) + b_2\int\gamma(n) = c_2$$

in which the a's, b's, and c's are constants which may be determined from the solution of the differential equations for the nth cycle. Assume that the circuit is oscillatory with the switch open or closed. (b) Solve for $\gamma(n)$ and $\rho(n)$, assuming that $\gamma(0)$ and $\rho(0)$ are zero.

9·11. In recording high surge voltages with a cathode-ray oscillograph a resistance potential divider is frequently used. It consists of a high resistance connected at one end to the high-potential source and at the other end to a short cable terminated at its far end in a resistance to ground. The oscillograph is connected across the latter resistance. When properly adjusted for no reflections, the cable and its termination may be represented by a resistance R_2 to ground at the low-potential end of the divider. The potential across R_2 is recorded by the oscillograph, but with a time delay caused by the cable.

When there are rapid changes in the applied surge voltage the voltage differences along the divider do not divide strictly proportional to the resistances because of

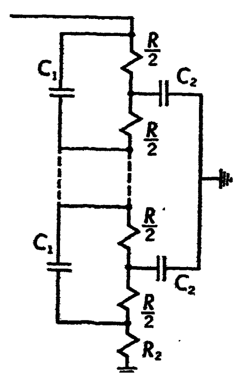


FIG. 9·P11

distributed capacitance along the resistor elements and between these elements and ground. Using lumped constants, an equivalent RC network may be made of N sections of the form shown in the figure.

If a step voltage of magnitude V is applied to the potential divider, find the voltage that appears across R_2. Consider only the first 5 microseconds. An approximate result may be obtained by neglecting R_2 in the calculation of the current to ground at the low-potential end and then multiplying this current by R_2. A result more nearly correct but more difficult to obtain can be found by including R_2 in the calculation of this current. The constants are

$$R/2 = 10^3 \text{ ohms}, \qquad R_2 = 40 \text{ ohms},$$

$$C_1 = 40 \times 10^{-6} \text{ microfarad}, \qquad N = 10.$$

$$C_2 = 10 \times 10^{-6} \text{ microfarad},$$

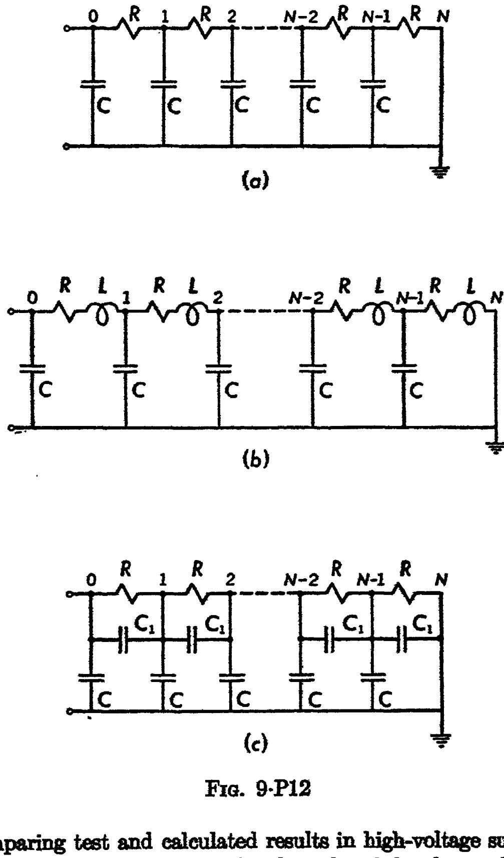


FIG. 9-P12

9-12. In comparing test and calculated results in high-voltage surge testing it is necessary to examine the effect on calculated results of the choice of equivalent network for the voltage divider. In the figure are shown three different equivalent networks for a certain resistance divider. Calculate the node voltage $v_{N-1}(t)$ in one

or more of these networks when the input is a step voltage of magnitude V. Consider only the first half microsecond. The constants are

$$R = 10^3 \text{ ohms}, \qquad L = 4 \times 10^{-3} \text{ millihenry},$$
$$C = 8 \times 10^{-6} \text{ microfarad}, \qquad N = 10.$$
$$C_1 = 30 \times 10^{-6} \text{ microfarad},$$

9·13. The characteristic equation of a system can be formed by equating to zero the determinant of the transform equations of this system. Consider a system composed of uniform sections so that its Nth-order determinant has the form

$$\begin{vmatrix} z_0 & z_1 & 0 & \cdots & 0 & 0 \\ z_1 & z_0 & z_1 & \cdots & 0 & 0 \\ 0 & z_1 & z_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z_0 & z_1 \\ 0 & 0 & 0 & \cdots & z_1 & z_0 \end{vmatrix} \triangleq D(s,N),$$

in which $z_0 \triangleq z_0(s)$ is the self-function for each of the uniform sections in the system and $z_1 \triangleq z_1(s)$ is the mutual function common to two adjacent uniform sections. By expanding this determinant in terms of its first minors it is seen that

$$D(s,n + 2) = z_0 D(s,n + 1) - z_1^2 D(s,n),$$

with $D(s,0) = 1$ and $D(s,1) = z_0$.

(a) Solve this difference equation for $D(s,n)$ and show that the characteristic equation of the system is satisfied if

$$\frac{z_0}{z_1} = 2 \cos \frac{k\pi}{N + 1}, \qquad k = 1, 2, \cdots, N.$$

From this set of equations the characteristic values can be found.

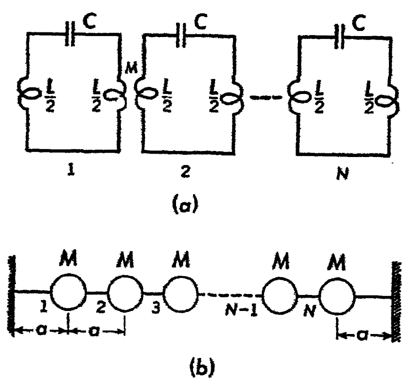


FIG. 9·P13

(b) By this method find the characteristic values of the systems shown in parts a and b of the figure. The system shown in part b represents a massless elastic string bearing N equally spaced equal masses which can vibrate laterally in a horizontal plane. For small displacements the tension T may be considered constant.

9·14. The dynamical system shown in the figure represents a 6-cylinder Diesel engine driving a flywheel and generator. The rotors are the equivalents of the rotating and reciprocating masses of the system, and the flywheel and generator have been combined.

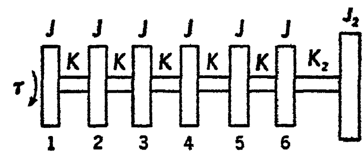


FIG. 9·P14

(a) If an external sinusoidal torque τ of angular frequency ω_1 acts on rotor 1, find the ratio of the torque in the shaft connecting rotors 4 and 5 to the amplitude of τ.

(b) What is the characteristic equation of the system?

(c) What are the characteristic values?

The constants in a single system of units are

$$J = 5 \times 10^3 \qquad K = 10^9 \qquad \omega_1 = 70.$$

$$J_2 = 10^5 \qquad K_2 = 5 \times 10^8$$

As an aid in the solution of this problem draw the mechanical network diagram for the system and from this form the analogous electric network with $i \sim \tau$. Divide this network into three parts: (1) a ladder-type network composed of identical T sections, (2) a voltage source and series termination at end 1, and (3) a shunt termination at end 2.

Appendix A

SUMMARY OF THEOREMS AND TRANSFORM PAIRS

A. BASIC $\mathfrak{L}$-TRANSFORMATION THEOREMS

DEFINITION. *A real*[1] *function* f(t) *which is defined and single valued almost everywhere for* $0 \leqq t$, *with* t *a real variable, and is such that the improper Lebesgue integral*

$$\int_0^\infty |f(t)|e^{-\sigma t}dt \triangleq \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_\epsilon^T |f(t)|e^{-\sigma t}dt < \infty \qquad [1]$$

for some real number σ, *will be called* $\mathfrak{L}$ *transformable.*

With any specified $f(t)$, the (greatest) lower bound of all the real numbers σ which satisfy condition 1 is called the *abscissa of absolute convergence* corresponding to that $f(t)$. It is denoted by σ_a.

THEOREM 1, $\mathfrak{L}$ TRANSFORMATION. *If* f(t) *is* $\mathfrak{L}$ *transformable, then the Laplace integral (improper Lebesgue integral)*

$$\int_0^\infty f(t)e^{-st}dt \triangleq \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_\epsilon^T f(t)e^{-st}dt, \qquad [2]$$

with s *a complex variable* $\sigma + j\omega$, *converges absolutely for* $\sigma_a < \sigma$ *to a function* F(s) *which is analytic in the half-plane* $\sigma_a < \sigma$.

This functional transformation is written in abbreviated notation as $\mathfrak{L}[f(t)] = F(s)$, $\sigma_a < \sigma$.

DEFINITION. *The inverse Laplace transformation* $\mathfrak{L}^{-1}$ *is defined implicitly by the relation*

$$\mathfrak{L}^{-1}\{\mathfrak{L}[f(t)]\} \;(=)\; f(t), \qquad 0 \leqq t. \qquad [3]$$

This can be written: If $F(s) = \mathfrak{L}[f(t)]$, then for $0 \leqq t$, $f(t) \;(=)\; \mathfrak{L}^{-1}[F(s)]$.

The $\mathfrak{L}^{-1}$ operation can be given an explicit representation in terms of known mathematical operations, as is shown by the following theorem.

THEOREM 2, $\mathfrak{L}^{-1}$ TRANSFORMATION. *If* F(s) *is the* $\mathfrak{L}$ *transform of a function* f(t), *then*

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{ts}ds \;(=)\; f(t), \qquad 0 \leqq t, \qquad [4]$$

in which $\sigma_a < c$.

[1] Complex functions of a real variable can be treated by resolving them into real and imaginary parts which are real.

THEOREM 3, UNICITY OF $\mathfrak{L}$ TRANSFORM. *If* f(t) *is* $\mathfrak{L}$ *transformable and* $\mathfrak{L}[f(t)] = F(s)$, $\sigma_a < \sigma$, *then its* $\mathfrak{L}$ *transform* F(s) *is unique.*

THEOREM 4, LEBESGUE UNICITY OF $\mathfrak{L}^{-1}$ TRANSFORM. *If* f(t) *is an* $\mathfrak{L}^{-1}$ *transform of* F(s), *then* f(t) *is Lebesgue unique, i.e., all other* $\mathfrak{L}^{-1}$ *transforms of* F(s) *are equal to* f(t) *almost everywhere for* $0 \leqq t$.

In symbols this can be written: $\mathfrak{L}^{-1}[F(s)] \;(=)\; f(t)$, $0 \leqq t$.

Theorems 5 to 21 expressed as operation-transform pairs are given in the first part of the table that follows.

B. OPERATION-TRANSFORM PAIRS

No.	Name	$f(t)$ $\quad 0 \leq t$	$F(s)$	Page
5	Linearity	$af(t)$ a is a constant or a variable independent of t and s. $f_1(t) \pm f_2(t)$	$aF(s)$ $F_1(s) \pm F_2(s)$	127
6	Real differentiation	$\dfrac{df(t)}{dt} \triangleq f'(t)$	$sF(s) - f(0+)$	127
6–a	Multiplication by s	$f'(t)$ if $f(0+) = 0$.	$sF(s)$	270
7	Real integration	$\int f(t)dt \triangleq f^{(-1)}(t)$	$\dfrac{F(s)}{s} + \dfrac{f^{(-1)}(0+)}{s}$	129
7–a	Division by s	$\int_0^t f(t)dt \triangleq f^{(-1)}(t) - f^{(-1)}(0+)$	$\dfrac{F(s)}{s}$	271
8	Scale change	$f\left(\dfrac{t}{a}\right)$ a is a positive constant or a positive variable independent of t and s.	$aF(as)$	226
9	Complex multiplication	$\int_0^t f_1(t-\tau)f_2(\tau)d\tau \triangleq f_1(t) * f_2(t)$	$F_1(s)F_2(s)$	228

10	Real translation	$f(t-a)$ if $f(t-a)=0,\ 0<t<a$ $f(t+a)$ if $f(t+a)=0,\ -a<t<0$ a is a non-negative real number.	$e^{-as}F(s)$ $e^{as}F(s)$	236
11	Complex translation	$e^{-at}f(t)$ $e^{at}f(t)$ a is a complex number with non-negative real part.	$F(s+a)$ $F(s-a)$	245
12	Second independent variable	$\lim_{a\to a_0} f(t,a)$ a is a second variable independent of t and s.	$\lim_{a\to a_0} F(s,a)$	252
13	Differentiation with respect to second independent variable	$\frac{\partial}{\partial a} f(t,a)$ a is a second variable independent of t and s.	$\frac{\partial}{\partial a} F(s,a)$	264
14	Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ if $sF(s)$ is analytic on the axis of imaginaries and in the right half-plane.		265
15	Initial value	$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$		267

B. OPERATION–TRANSFORM PAIRS — *continued*

No.	Name	$f(t) \qquad 0 \leq t$	$F(s)$	Page
16	Complex differentiation	$tf(t)$	$-\dfrac{d}{ds}F(s)$	272
17	Complex integration	$\dfrac{1}{t}f(t)$	$\displaystyle\int_s^\infty F(s)ds$	273
18	Integration with respect to second independent variable	$\displaystyle\int_{a_0}^{a} f(t,a)da$ a is a second variable independent of t and s.	$\displaystyle\int_{a_0}^{a} F(s,a)da$	274
19			$\displaystyle\frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty} F_1(s-w)F_2(w)dw \triangleq F_1(s) \circledast F_2(s),$ $\max(\sigma_{a_1}, \quad \sigma_{a_2}, \quad \sigma_{a_1}+\sigma_{a_2}) < \sigma, \quad \sigma_{a_2} < c_2 < \sigma - \sigma_{a_1}$	275
19–*a*	Real multiplication	$f_1(t)f_2(t)$	$\displaystyle\sum_{k=1}^{q} \frac{A_1(s_k)}{B_1'(s_k)} F_2(s-s_k)$ if $F_1(s) \triangleq \dfrac{A_1(s)}{B_1(s)}$ is rational algebraic fraction having only first-order poles.	277
19–*b*			$\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{m_k} \frac{(-1)^{m_k-j}K_{kj}}{(m_k-j)!}\left[\frac{d^{m_k-j}}{ds^{m_k-j}}F_2(s)\right]_{s=s-s_k}$ $K_{kj} \triangleq \dfrac{1}{(j-1)!}\left[\dfrac{d^{j-1}}{ds^{j-1}}(s-s_k)^{m_k}F_1(s)\right]_{s=s_k}$ if $F_1(s)$ is rational algebraic fraction having multiple-order poles.	278

20	Commutativity of $\mathfrak{L}$ with $\mathcal{R}$ and $\mathcal{I}$ transformations	$\mathcal{R}[f(t)]$ $\mathcal{I}[f(t)]$ $f(t)$ is a complex function	$\mathcal{R}[F(s)]$ $\mathcal{I}[F(s)]$	280
21	Translation of jump functions	$\mathcal{J}f(t+1)$	$e^s[F(s) - f(0)P(s)]$ $P(s) = \mathfrak{L}[p(t)]$ $p(t) \triangleq u(t) - u(t-1)$ is unit pulse.	289

C. FUNCTION-TRANSFORM PAIRS*

No.	$F(s)$	$f(t) \qquad 0 \leq t$
0.11	$\dfrac{A(s)}{B(s)}$ Rational proper fraction; first-order poles only.	$\displaystyle\sum_{k=1}^{q} \frac{A(s_k)}{B'(s_k)} e^{s_k t}$
0.21	$\dfrac{A(s)}{B(s)}$ Rational proper fraction; higher-order poles, general case.	$\displaystyle\sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{K_{kj}}{(m_k - j)!} t^{m_k - j} e^{s_k t}$ $\qquad m_1 + m_2 + \cdots + m_n = q$ $K_{kj} \triangleq \dfrac{1}{(j-1)!} \left[\dfrac{d^{j-1}}{ds^{j-1}} \dfrac{(s-s_k)^{m_k} A(s)}{B(s)} \right]_{s=s_k}$ $B(s) \triangleq (s-s_1)^{m_1}(s-s_2)^{m_2} \cdots (s-s_k)^{m_k} \cdots (s-s_n)^{m_n}$
1.01	1	$u_1(t) \triangleq \lim\limits_{a \to 0} \dfrac{u(t) - u(t-a)}{a}$, unit impulse at $t = 0$
1.02	s	$u_2(t) \triangleq \lim\limits_{a \to 0} \dfrac{u(t) - 2u(t-a) + u(t-2a)}{a^2}$, unit doublet impulse at $t = 0$
1.101	$\dfrac{1}{s}$	1, or $u(t)$, unit step at $t = 0$
1.102	$\dfrac{1}{s+\alpha}$	$e^{-\alpha t}$
1.105	$\dfrac{1}{(s+\alpha)(s+\gamma)}$	$\dfrac{e^{-\alpha t} - e^{-\gamma t}}{\gamma - \alpha}$
1.107	$\dfrac{s+a_0}{(s+\alpha)(s+\gamma)}$	$\dfrac{(a_0 - \alpha)e^{-\alpha t} - (a_0 - \gamma)e^{-\gamma t}}{\gamma - \alpha}$

* $\alpha, \beta, \gamma, \delta$ and λ are real numbers.

1.108	$\dfrac{1}{s(s+\alpha)(s+\gamma)}$	$\dfrac{1}{\alpha\gamma} + \dfrac{\gamma\epsilon^{-\alpha t} - \alpha\epsilon^{-\gamma t}}{\alpha\gamma(\alpha-\gamma)}$
1.109	$\dfrac{s+a_0}{s(s+\alpha)(s+\gamma)}$	$\dfrac{a_0}{\alpha\gamma} + \dfrac{a_0-\alpha}{\alpha(\alpha-\gamma)}\epsilon^{-\alpha t} + \dfrac{a_0-\gamma}{\gamma(\gamma-\alpha)}\epsilon^{-\gamma t}$
1.111	$\dfrac{s^2+a_1s+a_0}{s(s+\alpha)(s+\gamma)}$	$\dfrac{a_0}{\alpha\gamma} + \dfrac{\alpha^2-a_1\alpha+a_0}{\alpha(\alpha-\gamma)}\epsilon^{-\alpha t} - \dfrac{\gamma^2-a_1\gamma+a_0}{\gamma(\alpha-\gamma)}\epsilon^{-\gamma t}$
1.112	$\dfrac{1}{(s+\alpha)(s+\gamma)(s+\delta)}$	$\dfrac{\epsilon^{-\alpha t}}{(\gamma-\alpha)(\delta-\alpha)} + \dfrac{\epsilon^{-\gamma t}}{(\alpha-\gamma)(\delta-\gamma)} + \dfrac{\epsilon^{-\delta t}}{(\alpha-\delta)(\gamma-\delta)}$
1.114	$\dfrac{s+a_0}{(s+\alpha)(s+\gamma)(s+\delta)}$	$\dfrac{a_0-\alpha}{(\gamma-\alpha)(\delta-\alpha)}\epsilon^{-\alpha t} + \dfrac{a_0-\gamma}{(\alpha-\gamma)(\delta-\gamma)}\epsilon^{-\gamma t} + \dfrac{a_0-\delta}{(\alpha-\delta)(\gamma-\delta)}\epsilon^{-\delta t}$
1.118	$\dfrac{s^2+a_1s+a_0}{(s+\alpha)(s+\gamma)(s+\delta)}$	$\dfrac{\alpha^2-a_1\alpha+a_0}{(\gamma-\alpha)(\delta-\alpha)}\epsilon^{-\alpha t} + \dfrac{\gamma^2-a_1\gamma+a_0}{(\alpha-\gamma)(\delta-\gamma)}\epsilon^{-\gamma t} + \dfrac{\delta^2-a_1\delta+a_0}{(\alpha-\delta)(\gamma-\delta)}\epsilon^{-\delta t}$
1.201	$\dfrac{1}{s^2+\beta^2}$	$\dfrac{1}{\beta}\sin\beta t$
1.2011	$\dfrac{1}{s^2-\beta^2}$	$\dfrac{1}{\beta}\sinh\beta t$
1.202	$\dfrac{s}{s^2+\beta^2}$	$\cos\beta t$
1.2021	$\dfrac{s}{s^2-\beta^2}$	$\cosh\beta t$

C. FUNCTION-TRANSFORM PAIRS* — *continued*

No.	$F(s)$	$f(t) \qquad 0 \leqq t$
1.203	$\dfrac{s + a_0}{s^2 + \beta^2}$	$\dfrac{1}{\beta}(a_0 + \beta^2)^{1/2} \sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{\beta}{a_0}$
1.204	$\dfrac{1}{s(s^2 + \beta^2)}$	$\dfrac{1}{\beta^2}(1 - \cos \beta t)$
1.205	$\dfrac{s + a_0}{s(s^2 + \beta^2)}$	$\dfrac{a_0}{\beta^2} - \dfrac{(a_0^2 + \beta^2)^{1/2}}{\beta^2} \cos(\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{\beta}{a_0}$
1.207	$\dfrac{s^2 + a_1 s + a_0}{s(s^2 + \beta^2)}$	$\dfrac{a_0}{\beta^2} - \dfrac{[(a_0 - \beta^2)^2 + a_1^2\beta^2]^{1/2}}{\beta^2} \cos(\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{a_1\beta}{a_0 - \beta^2}$
1.210	$\dfrac{s + a_0}{(s + \alpha)(s^2 + \beta^2)}$	$\dfrac{a_0 - \alpha}{\alpha^2 + \beta^2} e^{-\alpha t} + \dfrac{1}{\beta}\left[\dfrac{a_0^2 + \beta^2}{\alpha^2 + \beta^2}\right]^{1/2} \sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{\beta}{a_0} - \tan^{-1} \dfrac{\beta}{\alpha}$
1.214	$\dfrac{s^2 + a_1 s + a_0}{(s + \alpha)(s^2 + \beta^2)}$	$\dfrac{\alpha^2 - a_1\alpha + a_0}{\alpha^2 + \beta^2} e^{-\alpha t} + \dfrac{1}{\beta}\left[\dfrac{(a_0 - \beta^2)^2 + a_1^2\beta^2}{\alpha^2 + \beta^2}\right]^{1/2} \sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{a_1\beta}{a_0 - \beta^2} - \tan^{-1} \dfrac{\beta}{\alpha}$

1.216	$\dfrac{s + a_0}{s(s+\alpha)(s^2+\beta^2)}$	$\dfrac{a_0}{\alpha\beta^2} + \dfrac{\alpha - a_0}{\alpha(\alpha^2+\beta^2)} e^{-\alpha t} - \dfrac{1}{\beta^2}\left[\dfrac{a_0^2+\beta^2}{\alpha^2+\beta^2}\right]^{1/2} \cos(\beta t + \psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0} - \tan^{-1}\dfrac{\beta}{\alpha}$
1.218	$\dfrac{s^2 + a_1 s + a_0}{s(s+\alpha)(s^2+\beta^2)}$	$\dfrac{a_0}{\alpha\beta^2} - \dfrac{\alpha^2 - a_1\alpha + a_0}{\alpha(\alpha^2+\beta^2)} e^{-\alpha t} - \dfrac{1}{\beta^2}\left[\dfrac{(a_0-\beta^2)^2 + a_1^2\beta^2}{\alpha^2+\beta^2}\right]^{1/2} \cos(\beta t + \psi)$ $\psi \triangleq \tan^{-1}\dfrac{a_1\beta}{a_0-\beta^2} - \tan^{-1}\dfrac{\beta}{\alpha}$
1.225	$\dfrac{s^2 + a_1 s + a_0}{(s+\alpha)(s+\gamma)(s^2+\beta^2)}$	$\dfrac{\alpha^2 - a_1\alpha + a_0}{(\gamma-\alpha)(\alpha^2+\beta^2)} e^{-\alpha t} + \dfrac{\gamma^2 - a_1\gamma + a_0}{(\alpha-\gamma)(\gamma^2+\beta^2)} e^{-\gamma t}$ $+ \dfrac{1}{\beta}\left[\dfrac{(a_0-\beta^2)^2 + a_1^2\beta^2}{(\alpha^2+\beta^2)(\gamma^2+\beta^2)}\right]^{1/2} \sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1}\dfrac{a_1\beta}{a_0-\beta^2} - \tan^{-1}\dfrac{\beta}{\alpha} - \tan^{-1}\dfrac{\beta}{\gamma}$
1.229	$\dfrac{s^3 + a_2 s^2 + a_1 s + a_0}{(s+\alpha)(s+\gamma)(s^2+\beta^2)}$	$\dfrac{-\alpha^3 + a_2\alpha^2 - a_1\alpha + a_0}{(\gamma-\alpha)(\alpha^2+\beta^2)} e^{-\alpha t} + \dfrac{-\gamma^3 + a_2\gamma^2 - a_1\gamma + a_0}{(\alpha-\gamma)(\gamma^2+\beta^2)} e^{-\gamma t}$ $+ \dfrac{1}{\beta}\left[\dfrac{(a_0 - a_2\beta^2)^2 + \beta^2(a_1-\beta^2)^2}{(\alpha^2+\beta^2)(\gamma^2+\beta^2)}\right]^{1/2} \sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta(a_1-\beta^2)}{a_0 - a_2\beta^2} - \tan^{-1}\dfrac{\beta}{\alpha} - \tan^{-1}\dfrac{\beta}{\gamma}$
1.253	$\dfrac{s}{(s^2+\beta^2)(s^2+\lambda^2)}$	$\dfrac{\cos\beta t - \cos\lambda t}{\lambda^2 - \beta^2}$
1.253-a	$\dfrac{s}{[s^2+(\beta+\lambda)^2][s^2+(\beta-\lambda)^2]}$	$\dfrac{1}{2\lambda\beta}\sin\lambda t \cdot \sin\beta t$

C. FUNCTION-TRANSFORM PAIRS*—continued

No.	$F(s)$	$f(t) \qquad 0 \leq t$
1.258	$\dfrac{s^2 + a_1 s + a_0}{(s^2+\beta^2)(s^2+\lambda^2)}$	$\dfrac{[(a_0-\beta^2)^2 + a_1^2\beta^2]^{1/2}}{\beta(\lambda^2-\beta^2)} \sin(\beta t + \psi_1) + \dfrac{[(a_0-\lambda^2)^2 + a_1^2\lambda^2]^{1/2}}{\lambda(\beta^2-\lambda^2)} \sin(\lambda t + \psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{a_1\beta}{a_0-\beta^2}; \quad \psi_2 \triangleq \tan^{-1}\dfrac{a_1\lambda}{a_0-\lambda^2}$
1.262	$\dfrac{s^3 + a_2 s^2 + a_1 s + a_0}{(s^2+\beta^2)(s^2+\lambda^2)}$	$\dfrac{[(a_0-a_2\beta^2)^2 + \beta^2(a_1-\beta^2)^2]^{1/2}}{\beta(\lambda^2-\beta^2)} \sin(\beta t + \psi_1)$ $+ \dfrac{[(a_0-a_2\lambda^2)^2 + \lambda^2(a_1-\lambda^2)^2]^{1/2}}{\lambda(\beta^2-\lambda^2)} \sin(\lambda t + \psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{\beta(a_1-\beta^2)}{a_0-a_2\beta^2}; \quad \psi_2 \triangleq \tan^{-1}\dfrac{\lambda(a_1-\lambda^2)}{a_0-a_2\lambda^2}$
1.301	$\dfrac{1}{(s+\alpha)^2+\beta^2}$	$\dfrac{1}{\beta} e^{-\alpha t} \sin \beta t$
1.303	$\dfrac{s+a_0}{(s+\alpha)^2+\beta^2}$	$\dfrac{1}{\beta}[(a_0-\alpha)^2+\beta^2]^{1/2} e^{-\alpha t} \sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0-\alpha}$
1.3031	$\dfrac{s+\alpha}{(s+\alpha)^2+\beta^2}$	$e^{-\alpha t}\cos \beta t$
1.304	$\dfrac{1}{s[(s+\alpha)^2+\beta^2]}$	$\dfrac{1}{\beta_0^2} + \dfrac{1}{\beta_0\beta} e^{-\alpha t}\sin(\beta t - \psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{-\alpha}$ $\beta_0^2 \triangleq \alpha^2+\beta^2$

1.305	$\dfrac{s+a_0}{s[(s+\alpha)^2+\beta^2]}$	$\dfrac{a_0}{\beta_0^2}+\dfrac{1}{\beta\beta_0}[(a_0-\alpha)^2+\beta^2]^{1/2}e^{-\alpha t}\sin(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0-\alpha}-\tan^{-1}\dfrac{\beta}{-\alpha}$ $\beta_0^2 \triangleq \alpha^2+\beta^2$
1.307	$\dfrac{s^2+a_1s+a_0}{s[(s+\alpha)^2+\beta^2]}$	$\dfrac{a_0}{\beta_0^2}+\dfrac{1}{\beta\beta_0}[(\alpha^2-\beta^2-a_1\alpha+a_0)^2+\beta^2(a_1-2\alpha)^2]^{1/2}e^{-\alpha t}\sin(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta(a_1-2\alpha)}{\alpha^2-\beta^2-a_1\alpha+a_0}-\tan^{-1}\dfrac{\beta}{-\alpha}$ $\beta_0^2 = \beta^2+\alpha^2$
1.308	$\dfrac{1}{(s+\gamma)[(s+\alpha)^2+\beta^2]}$	$\dfrac{e^{-\gamma t}}{(\gamma-\alpha)^2+\beta^2}+\dfrac{1}{\beta[(\gamma-\alpha)^2+\beta^2]^{1/2}}e^{-\alpha t}\sin(\beta t-\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{\gamma-\alpha}$
1.310	$\dfrac{s+a_0}{(s+\gamma)[(s+\alpha)^2+\beta^2]}$	$\dfrac{a_0-\gamma}{(\alpha-\gamma)^2+\beta^2}e^{-\gamma t}+\dfrac{1}{\beta}\left[\dfrac{(a_0-\alpha)^2+\beta^2}{(\gamma-\alpha)^2+\beta^2}\right]^{1/2}e^{-\alpha t}\sin(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0-\alpha}-\tan^{-1}\dfrac{\beta}{\gamma-\alpha}$
1.314	$\dfrac{s^2+a_1s+a_0}{(s+\gamma)[(s+\alpha)^2+\beta^2]}$	$\dfrac{\gamma^2-a_1\gamma+a_0}{(\alpha-\gamma)^2+\beta^2}e^{-\gamma t}+\dfrac{1}{\beta}\left[\dfrac{(\alpha^2-\beta^2-a_1\alpha+a_0)^2+\beta^2(a_1-2\alpha)^2}{(\gamma-\alpha)^2+\beta^2}\right]^{1/2}e^{-\alpha t}\sin(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta(a_1-2\alpha)}{\alpha^2-\beta^2-a_1\alpha+a_0}-\tan^{-1}\dfrac{\beta}{\gamma-\alpha}$
1.319	$\dfrac{1}{s(s+\gamma)[(s+\alpha)^2+\beta^2]}$	$\dfrac{1}{\gamma\beta_0^2}-\dfrac{1}{\gamma[(\alpha-\gamma)^2+\beta^2]}e^{-\gamma t}+\dfrac{1}{\beta\beta_0[(\gamma-\alpha)^2+\beta^2]^{1/2}}e^{-\alpha t}\sin(\beta t-\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{-\alpha}+\tan^{-1}\dfrac{\beta}{\gamma-\alpha}$ $\beta_0^2 \triangleq \alpha^2+\beta^2$

C. FUNCTION-TRANSFORM PAIRS* — *continued*

No.	$F(s)$	$f(t) \qquad 0 \le t$
1.320	$\dfrac{s + a_0}{s(s+\gamma)[(s+\alpha)^2+\beta^2]}$	$\dfrac{a_0}{\gamma\beta_0^2} + \dfrac{\gamma - a_0}{\gamma[(\alpha-\gamma)^2+\beta^2]} e^{-\gamma t} + \dfrac{1}{\beta\beta_0}\left[\dfrac{(a_0-\alpha)^2+\beta^2}{(\gamma-\alpha)^2+\beta^2}\right]^{1/2} e^{-\alpha t}\sin(\beta t + \psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0-\alpha} - \tan^{-1}\dfrac{\beta}{\gamma-\alpha} - \tan^{-1}\dfrac{\beta}{-\alpha}$ $\beta_0^2 \triangleq \alpha^2+\beta^2$
1.330	$\dfrac{s^2 + a_1 s + a_0}{(s+\gamma)(s+\delta)[(s+\alpha)^2+\beta^2]}$	$\dfrac{\gamma^2 - a_1\gamma + a_0}{(\delta-\gamma)[(\alpha-\gamma)^2+\beta^2]} e^{-\gamma t} + \dfrac{\delta^2 - a_1\delta + a_0}{(\gamma-\delta)[(\alpha-\delta)^2+\beta^2]} e^{-\delta t}$ $+ \dfrac{1}{\beta}\left\{\dfrac{(\alpha^2-\beta^2-a_1\alpha+a_0)^2+\beta^2(a_1-2\alpha)^2}{[(\delta-\alpha)^2+\beta^2][(\gamma-\alpha)^2+\beta^2]}\right\}^{1/2} e^{-\alpha t}\sin(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta(a_1-2\alpha)}{\alpha^2-\beta^2-a_1\alpha+a_0} - \tan^{-1}\dfrac{\beta}{\gamma-\alpha} - \tan^{-1}\dfrac{\beta}{\delta-\alpha}$
1.357	$\dfrac{1}{(s^2+\lambda^2)[(s+\alpha)^2+\beta^2]}$	$\dfrac{1}{[(\beta_0^2-\lambda^2)^2+4\alpha^2\lambda^2]^{1/2}}\left[\dfrac{1}{\lambda}\sin(\lambda t-\psi_1) + \dfrac{1}{\beta}e^{-\alpha t}\sin(\beta t-\psi_2)\right]$ $\psi_1 \triangleq \tan^{-1}\dfrac{2\alpha\lambda}{\beta_0^2-\lambda^2};\quad \psi_2 \triangleq \dfrac{-2\alpha\beta}{\alpha^2-\beta^2+\lambda^2};\quad \beta_0^2 \triangleq \alpha^2+\beta^2$
1.359	$\dfrac{s + a_0}{(s^2+\lambda^2)[(s+\alpha)^2+\beta^2]}$	$\dfrac{1}{\lambda}\left[\dfrac{a_0^2+\lambda^2}{(\beta_0^2-\lambda^2)^2+4\alpha^2\lambda^2}\right]^{1/2}\sin(\lambda t+\psi_1)$ $+ \dfrac{1}{\beta}\left[\dfrac{(a_0-\alpha)^2+\beta^2}{(\beta_0^2-\lambda^2)^2+4\alpha^2\lambda^2}\right]^{1/2} e^{-\alpha t}\sin(\beta t+\psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{\lambda}{a_0} - \tan^{-1}\dfrac{2\alpha\lambda}{\beta_0^2-\lambda^2};\quad \psi_2 \triangleq \tan^{-1}\dfrac{\beta}{a_0-\alpha} - \tan^{-1}\dfrac{-2\alpha\beta}{\alpha^2-\beta^2+\lambda^2}$ $\beta_0^2 \triangleq \alpha^2+\beta^2$

1.363	$\dfrac{s^2 + a_1 s + a_0}{(s^2 + \lambda^2)[(s+\alpha)^2 + \beta^2]}$	$\dfrac{1}{\lambda}\left[\dfrac{(a_0 - \lambda^2)^2 + a_1^2\lambda^2}{(\beta_0^2 - \lambda^2)^2 + 4\alpha^2\lambda^2}\right]^{1/2} \sin(\lambda t + \psi_1)$ $+ \dfrac{1}{\beta}\left[\dfrac{(\alpha^2 - \beta^2 - a_1\alpha + a_0)^2 + \beta^2(a_1 - 2\alpha)^2}{(\beta_0^2 - \lambda^2)^2 + 4\alpha^2\lambda^2}\right]^{1/2} e^{-\alpha t} \sin(\beta t + \psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{a_1\lambda}{a_0 - \lambda^2} - \tan^{-1}\dfrac{2\alpha\lambda}{\beta_0^2 - \lambda^2}$ $\psi_2 \triangleq \tan^{-1}\dfrac{\beta(a_1 - 2\alpha)}{\alpha^2 - \beta^2 - a_1\alpha + a_0} - \tan^{-1}\dfrac{-2\alpha\beta}{\alpha^2 - \beta^2 + \lambda^2}$ $\beta_0^2 \triangleq \alpha^2 + \beta^2$
1.376	$\dfrac{s + a_0}{(s+\gamma)(s^2 + \lambda^2)[(s+\alpha)^2 + \beta^2]}$	$\dfrac{a_0 - \gamma}{(\lambda^2 + \gamma^2)[(\alpha - \gamma)^2 + \beta^2]} e^{-\gamma t} + \dfrac{1}{\lambda}\left\{\dfrac{a_0^2 + \lambda^2}{(\gamma^2 + \lambda^2)[(\beta_0^2 - \lambda^2)^2 + 4\alpha^2\lambda^2]}\right\}^{1/2} \sin(\lambda t + \psi_1)$ $+ \dfrac{1}{\beta}\left\{\dfrac{(a_0 - \alpha)^2 + \beta^2}{[(\gamma - \alpha)^2 + \beta^2][(\beta_0^2 - \lambda^2)^2 + 4\alpha^2\lambda^2]}\right\}^{1/2} e^{-\alpha t} \sin(\beta t + \psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{\lambda}{a_0} - \tan^{-1}\dfrac{\lambda}{\gamma} - \tan^{-1}\dfrac{2\alpha\lambda}{\beta_0^2 - \lambda^2}$ $\psi_2 \triangleq \tan^{-1}\dfrac{\beta}{a_0 - \alpha} - \tan^{-1}\dfrac{\beta}{\gamma - \alpha} - \tan^{-1}\dfrac{-2\alpha\beta}{\alpha^2 - \beta^2 + \lambda^2}$ $\beta_0^2 \triangleq \alpha^2 + \beta^2$
2.101	$\dfrac{1}{s^2}$	t
2.102	$\dfrac{1}{s^n}$	$\dfrac{1}{(n-1)!} t^{n-1}$ n is a positive integer.
2.103	$\dfrac{1}{(s+\alpha)s^2}$	$\dfrac{e^{-\alpha t} + \alpha t - 1}{\alpha^2}$

C. FUNCTION-TRANSFORM PAIRS* —*continued*

No.	$F(s)$	$f(t)$ $\qquad 0 \leqq t$	
2.104	$\dfrac{s + a_0}{(s+\alpha)s^2}$	$\dfrac{a_0 - \alpha}{\alpha^2} e^{-\alpha t} + \dfrac{a_0}{\alpha} t + \dfrac{\alpha - a_0}{\alpha^2}$	
2.106	$\dfrac{s^2 + a_1 s + a_0}{(s+\alpha)s^2}$	$\dfrac{\alpha^2 - a_1\alpha + a_0}{\alpha^2} e^{-\alpha t} + \dfrac{a_0}{\alpha} t + \dfrac{a_1\alpha - a_0}{\alpha^2}$	
2.118	$\dfrac{1}{(s+\alpha)^2}$	$te^{-\alpha t}$	
2.120	$\dfrac{s + a_0}{(s+\alpha)^2}$	$[(a_0 - \alpha)t + 1]e^{-\alpha t}$	
2.121	$\dfrac{1}{(s+\alpha)^n}$	$\dfrac{1}{(n-1)!} t^{n-1} e^{-\alpha t}$	n is a positive integer.
2.1362	$\dfrac{s^n}{(s+\alpha)^{n+1}}$	$e^{-\alpha t} \displaystyle\sum_{k=0}^{n} \dfrac{n!(-\alpha)^k}{(n-k)!(k!)^2} t^k$	n is a non-negative integer.
2.137	$\dfrac{1}{s(s+\alpha)^2}$	$\dfrac{1 - (1 + \alpha t)e^{-\alpha t}}{\alpha^2}$	
2.138	$\dfrac{s + a_0}{s(s+\alpha)^2}$	$\dfrac{a_0}{\alpha^2} + \left(\dfrac{\alpha - a_0}{\alpha} t - \dfrac{a_0}{\alpha^2}\right) e^{-\alpha t}$	
2.140	$\dfrac{s^2 + a_1 s + a_0}{s(s+\alpha)^2}$	$\dfrac{a_0}{\alpha^2} + \left(\dfrac{a_1\alpha - a_0 - \alpha^2}{\alpha} t + \dfrac{\alpha^2 - a_0}{\alpha^2}\right) e^{-\alpha t}$	
2.152	$\dfrac{1}{(s+\gamma)(s+\alpha)^2}$	$\dfrac{1}{(\gamma - \alpha)^2} e^{-\gamma t} + \dfrac{(\gamma - \alpha)t - 1}{(\gamma - \alpha)^2} e^{-\alpha t}$	

2.154	$\dfrac{s+a_0}{(s+\gamma)(s+\alpha)^2}$	$\dfrac{a_0-\gamma}{(\alpha-\gamma)^2}\epsilon^{-\gamma t}+\left[\dfrac{a_0-\alpha}{\gamma-\alpha}t+\dfrac{\gamma-a_0}{(\gamma-\alpha)^2}\right]\epsilon^{-\alpha t}$
2.158	$\dfrac{s^2+a_1s+a_0}{(s+\gamma)(s+\alpha)^2}$	$\dfrac{\gamma^2-a_1\gamma+a_0}{(\alpha-\gamma)^2}\epsilon^{-\gamma t}+\left[\dfrac{\alpha^2-a_1\alpha+a_0}{\gamma-\alpha}t+\dfrac{\alpha^2-2\alpha\gamma+a_1\gamma-a_0}{(\gamma-\alpha)^2}\right]\epsilon^{-\alpha t}$
2.161	$\dfrac{s+a_0}{(s+\gamma)(s+\alpha)^3}$	$\dfrac{a_0-\gamma}{(\alpha-\gamma)^3}\epsilon^{-\gamma t}+\left[\dfrac{a_0-\alpha}{2(\gamma-\alpha)}t^2+\dfrac{\gamma-a_0}{(\gamma-\alpha)^2}t+\dfrac{a_0-\gamma}{(\gamma-\alpha)^3}\right]\epsilon^{-\alpha t}$
2.187	$\dfrac{s+a_0}{s(s+\gamma)(s+\alpha)^2}$	$\dfrac{a_0}{\gamma\alpha^2}+\dfrac{\gamma-a_0}{\gamma(\alpha-\gamma)^2}\epsilon^{-\gamma t}+\left[\dfrac{a_0-\alpha}{\alpha(\alpha-\gamma)}t+\dfrac{2a_0\alpha-\alpha^2-a_0\gamma}{\alpha^2(\alpha-\gamma)^2}\right]\epsilon^{-\alpha t}$
2.189	$\dfrac{s^2+a_1s+a_0}{s(s+\gamma)(s+\alpha)^2}$	$\dfrac{a_0}{\gamma\alpha^2}-\dfrac{\gamma^2-a_1\gamma+a_0}{\gamma(\alpha-\gamma)^2}\epsilon^{-\gamma t}+\left[\dfrac{\alpha^2-a_1\alpha+a_0}{\alpha(\alpha-\gamma)}t+\dfrac{(\gamma-a_1)\alpha^2+(2\alpha-\gamma)a_0}{\alpha^2(\alpha-\gamma)^2}\right]\epsilon^{-\alpha t}$
2.199	$\dfrac{s+a_0}{(s+\gamma)(s+\delta)(s+\alpha)^2}$	$\dfrac{a_0-\gamma}{(\delta-\gamma)(\alpha-\gamma)^2}\epsilon^{-\gamma t}+\dfrac{a_0-\delta}{(\gamma-\delta)(\alpha-\delta)^2}\epsilon^{-\delta t}$ $+\left[\dfrac{a_0-\alpha}{(\gamma-\alpha)(\delta-\alpha)}t+\dfrac{2a_0\alpha-\alpha^2-a_0(\gamma+\delta)+\gamma\delta}{(\gamma-\alpha)^2(\delta-\alpha)^2}\right]\epsilon^{-\alpha t}$
2.247	$\dfrac{s+a_0}{(s+\alpha)(s+\gamma)s^2}$	$\dfrac{a_0-\alpha}{\alpha^2(\gamma-\alpha)}\epsilon^{-\alpha t}+\dfrac{a_0-\gamma}{\gamma^2(\alpha-\gamma)}\epsilon^{-\gamma t}+\dfrac{a_0}{\alpha\gamma}t+\dfrac{\alpha\gamma-a_0(\alpha+\gamma)}{\alpha^2\gamma^2}$
2.249	$\dfrac{s^2+a_1s+a_0}{(s+\alpha)(s+\gamma)s^2}$	$\dfrac{\alpha^2-a_1\alpha+a_0}{\alpha^2(\gamma-\alpha)}\epsilon^{-\alpha t}+\dfrac{\gamma^2-a_1\gamma+a_0}{\gamma^2(\alpha-\gamma)}\epsilon^{-\gamma t}+\dfrac{a_0}{\alpha\gamma}t+\dfrac{a_1\alpha\gamma-a_0(\alpha+\gamma)}{\alpha^2\gamma^2}$
2.276	$\dfrac{s^2+a_1s+a_0}{(s+\alpha)^2s^2}$	$\left[\dfrac{\alpha^2-a_1\alpha+a_0}{\alpha^2}t+\dfrac{2a_0-a_1\alpha}{\alpha^3}\right]\epsilon^{-\alpha t}+\dfrac{a_0}{\alpha^2}t+\dfrac{a_1\alpha-2a_0}{\alpha^3}$
2.292	$\dfrac{s+a_0}{(s+\alpha)^2(s+\gamma)^2}$	$\left[\dfrac{a_0-\alpha}{(\gamma-\alpha)^2}t+\dfrac{\alpha+\gamma-2a_0}{(\gamma-\alpha)^3}\right]\epsilon^{-\alpha t}+\left[\dfrac{a_0-\gamma}{(\alpha-\gamma)^2}t+\dfrac{\alpha+\gamma-2a_0}{(\alpha-\gamma)^3}\right]\epsilon^{-\gamma t}$

C. FUNCTION-TRANSFORM PAIRS* — *continued*

No.	$F(s)$	$f(t)$ $\quad 0 \leq t$
2.296	$\dfrac{s^2 + a_1 s + a_0}{(s+\alpha)^2(s+\gamma)^2}$	$\left[\dfrac{\alpha^2 - a_1\alpha + a_0}{(\gamma-\alpha)^2} t + \dfrac{a_1(\alpha+\gamma) - 2(\alpha\gamma + a_0)}{(\gamma-\alpha)^3}\right] e^{-\alpha t} + \left[\dfrac{\gamma^2 - a_1\gamma + a_0}{(\gamma-\alpha)^2} t - \dfrac{a_1(\alpha+\gamma) - 2(\alpha\gamma + a_0)}{(\gamma-\alpha)^3}\right] e^{-\gamma t}$
2.342	$\dfrac{s^2 + a_1 s + a_0}{(s+\alpha)^3 s^2}$	$\left(\dfrac{\alpha^2 - a_1\alpha + a_0}{2\alpha^2} t^2 + \dfrac{-a_1\alpha + 2a_0}{\alpha^3} t + \dfrac{-a_1\alpha + 3a_0}{\alpha^4}\right) e^{-\alpha t} + \dfrac{a_0}{\alpha^3} t + \dfrac{a_1\alpha - 3a_0}{\alpha^4}$
2.401	$\dfrac{1}{(s^2+\beta^2)s^2}$	$\dfrac{1}{\beta^2} t - \dfrac{1}{\beta^3} \sin \beta t$
2.4011	$\dfrac{1}{(s^2-\beta^2)s^2}$	$\dfrac{1}{\beta^3} \sinh \beta t - \dfrac{1}{\beta^2} t$
2.402	$\dfrac{s + a_0}{(s^2+\beta^2)s^2}$	$\dfrac{a_0}{\beta^2} t + \dfrac{1}{\beta^2} - \dfrac{1}{\beta^3} (a_0^2 + \beta^2)^{1/2} \sin (\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{\beta}{a_0}$
2.404	$\dfrac{s^2 + a_1 s + a_0}{(s^2+\beta^2)s^2}$	$\dfrac{a_0}{\beta^2} t + \dfrac{a_1}{\beta^2} - \dfrac{1}{\beta^3} [(a_0 - \beta^2)^2 + a_1^2\beta^2]^{1/2} \sin (\beta t + \psi)$ $\psi \triangleq \tan^{-1} \dfrac{a_1\beta}{a_0 - \beta^2}$
2.406	$\dfrac{1}{(s^2+\beta^2)s^3}$	$\dfrac{1}{\beta^4} (\cos \beta t - 1) + \dfrac{1}{2\beta^2} t^2$
2.4061	$\dfrac{1}{(s^2-\beta^2)s^3}$	$\dfrac{1}{\beta^4} (\cosh \beta t - 1) - \dfrac{1}{2\beta^2} t^2$

2.418	$\dfrac{1}{(s^2+\beta^2)(s+\alpha)^2}$	$\dfrac{1}{\beta(\alpha^2+\beta^2)}\sin(\beta t-\psi)+\left[\dfrac{1}{\alpha^2+\beta^2}t+\dfrac{2\alpha}{(\alpha^2+\beta^2)^2}\right]e^{-\alpha t}$ $\psi \triangleq 2\tan^{-1}\dfrac{\beta}{\alpha}$
2.420	$\dfrac{s+a_0}{(s^2+\beta^2)(s+\alpha)^2}$	$\dfrac{(a_0^2+\beta^2)^{1/2}}{\beta(\alpha^2+\beta^2)}\sin(\beta t+\psi)+\left[\dfrac{a_0-\alpha}{\alpha^2+\beta^2}t+\dfrac{2a_0\alpha+\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2}\right]e^{-\alpha t}$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0}-2\tan^{-1}\dfrac{\beta}{\alpha}$
2.424	$\dfrac{s^2+a_1s+a_0}{(s^2+\beta^2)(s+\alpha)^2}$	$\dfrac{[(a_0-\beta^2)^2+a_1^2\beta^2]^{1/2}}{\beta(\alpha^2+\beta^2)}\sin(\beta t+\psi)$ $+\left[\dfrac{\alpha^2-a_1\alpha+a_0}{\alpha^2+\beta^2}t+\dfrac{a_1(\beta^2-\alpha^2)+2\alpha(a_0-\beta^2)}{(\alpha^2+\beta^2)^2}\right]e^{-\alpha t}$ $\psi \triangleq \tan^{-1}\dfrac{a_1\beta}{a_0-\beta^2}-2\tan^{-1}\dfrac{\beta}{\alpha}$
2.446	$\dfrac{s+a_0}{s(s^2+\beta^2)(s+\alpha)^2}$	$\dfrac{a_0}{\beta^2\alpha^2}-\dfrac{(a_0^2+\beta^2)^{1/2}}{\beta^2(\alpha^2+\beta^2)}\cos(\beta t+\psi)+\left[\dfrac{\alpha-a_0}{\alpha(\alpha^2+\beta^2)}t+\dfrac{2\alpha^3-3a_0\alpha^2-a_0\beta^2}{\alpha^2(\alpha^2+\beta^2)^2}\right]e^{-\alpha t}$ $\psi \triangleq \tan^{-1}\dfrac{\beta}{a_0}-2\tan^{-1}\dfrac{\beta}{\alpha}$
2.457	$\dfrac{s^2+a_1s+a_0}{(s+\gamma)(s^2+\beta^2)(s+\alpha)^2}$	$\dfrac{\gamma^2-a_1\gamma+a_0}{(\gamma^2+\beta^2)(\alpha-\gamma)^2}e^{-\gamma t}+\dfrac{[(a_0-\beta^2)^2+a_1^2\beta^2]^{1/2}}{\beta(\gamma^2+\beta^2)^{1/2}(\alpha^2+\beta^2)}\sin(\beta t+\psi)+\dfrac{\alpha^2-a_1\alpha+a_0}{(\gamma-\alpha)(\alpha^2+\beta^2)}te^{-\alpha t}$ $+\dfrac{(\gamma-\alpha)(\alpha^2+\beta^2)(a_1-2\alpha)-(\alpha^2-a_1\alpha+a_0)(3\alpha^2+\beta^2-2\alpha\gamma)}{(\gamma-\alpha)^2(\alpha^2+\beta^2)^2}e^{-\alpha t}$ $\psi \triangleq \tan^{-1}\dfrac{a_1\beta}{a_0-\beta^2}-\tan^{-1}\dfrac{\beta}{\gamma}-2\tan^{-1}\dfrac{\beta}{\alpha}$

C. FUNCTION-TRANSFORM PAIRS* —*continued*

No.	$F(s)$	$f(t) \qquad 0 \leqq t$
2.501	$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin \beta t - \beta t \cos \beta t)$
2.502	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta} t \sin \beta t$
2.504	$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin \beta t + \beta t \cos \beta t)$
2.5061	$\dfrac{s^2-\beta^2}{(s^2+\beta^2)^2}$	$t \cos \beta t$
2.512	$\dfrac{1}{s(s^2+\beta^2)^2}$	$\dfrac{1}{\beta^4}(1-\cos \beta t) - \dfrac{1}{2\beta^3} t \sin \beta t$
2.515	$\dfrac{s^2+a_1s+a_0}{s(s^2+\beta^2)^2}$	$\dfrac{a_0}{\beta^4} - \dfrac{[(a_0-\beta^2)^2+a_1^2\beta^2]^{1/2}}{2\beta^3} t \sin(\beta t+\psi_1) - \dfrac{(4a_0^2+a_1^2\beta^2)^{1/2}}{2\beta^4}\cos(\beta t+\psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{a_1\beta}{a_0-\beta^2}$ $\psi_2 \triangleq \tan^{-1}\dfrac{a_1\beta}{2a_0}$
2.601	$\dfrac{1}{[(s+\alpha)^2+\beta^2]s^2}$	$\dfrac{1}{\beta_0^2}\left[t - \dfrac{2\alpha}{\beta_0^2} + \dfrac{1}{\beta}e^{-\alpha t}\sin(\beta t-\psi)\right]$ $\psi \triangleq 2\tan^{-1}\dfrac{\beta}{-\alpha}$ $\beta_0^2 \triangleq \alpha^2+\beta^2$

2.618	$\dfrac{1}{(s+\gamma)^2[(s+\alpha)^2+\beta^2]}$	$\dfrac{1}{(\alpha-\gamma)^2+\beta^2}\left[te^{-\gamma t}+\dfrac{2(\gamma-\alpha)}{(\alpha-\gamma)^2+\beta^2}e^{-\gamma t}+\dfrac{1}{\beta}e^{-\alpha t}\sin(\beta t-\psi)\right]$ $\psi \triangleq 2\tan^{-1}\dfrac{\beta}{\gamma-\alpha}$
2.624	$\dfrac{s^2+a_1 s+a_0}{(s+\gamma)^2[(s+\alpha)^2+\beta^2]}$	$\dfrac{\gamma^2-a_1\gamma+a_0}{(\alpha-\gamma)^2+\beta^2}te^{-\gamma t}+\dfrac{[(\alpha-\gamma)^2+\beta^2](a_1-2\gamma)-2(\alpha-\gamma)(\gamma^2-a_1\gamma+a_0)}{[(\alpha-\gamma)^2+\beta^2]^2}e^{-\gamma t}$ $+\dfrac{[(\alpha^2-\beta^2-a_1\alpha+a_0)^2+\beta^2(a_1-2\alpha)^2]^{1/2}}{\beta[(\gamma-\alpha)^2+\beta^2]}e^{-\alpha t}\sin(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{\beta(a_1-2\alpha)}{\alpha^2-\beta^2-a_1\alpha+a_0}-2\tan^{-1}\dfrac{\beta}{\gamma-\alpha}$
2.701	$\dfrac{1}{[(s+\alpha)^2+\beta^2]^2}$	$\dfrac{1}{2\beta^3}e^{-\alpha t}(\sin\beta t-\beta t\cos\beta t)$
2.7031	$\dfrac{s+\alpha}{[(s+\alpha)^2+\beta^2]^2}$	$\dfrac{1}{2\beta}te^{-\alpha t}\sin\beta t$
2.706	$\dfrac{s^2+a_0}{[(s+\alpha)^2+\beta^2]^2}$	$\dfrac{\beta_0^2+a_0}{2\beta^3}e^{-\alpha t}\sin\beta t-\dfrac{[(\alpha^2-\beta^2+a_0)^2+4\alpha^2\beta^2]^{1/2}}{2\beta^2}te^{-\alpha t}\cos(\beta t+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{-2\alpha\beta}{\alpha^2-\beta^2+a_0};\quad \beta_0^2 \triangleq \alpha^2+\beta^2$
2.7071	$\dfrac{(s+\alpha)^2-\beta^2}{[(s+\alpha)^2+\beta^2]^2}$	$te^{-\alpha t}\cos\beta t$
3.01	$\tan^{-1}\dfrac{\beta}{s}$	$\dfrac{\sin\beta t}{t}$
3.02	$\ln\dfrac{s+\beta}{s+\alpha}$	$\dfrac{e^{-\alpha t}-e^{-\beta t}}{t}$

C. FUNCTION-TRANSFORM PAIRS* — *continued*

No.	$F(s)$	$f(t)$ $\quad 0 \leqq t$
3.05	$e^{s^2/4a} \operatorname{cerf} \dfrac{s}{2\sqrt{a}}$	$2\sqrt{\dfrac{a}{\pi}}\, e^{-at^2}$ $\operatorname{cerf} y \triangleq 1 - \operatorname{erf} y \triangleq 1 - \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^y e^{-x^2} dx$
5.01	$\dfrac{1}{\sqrt{s^2 + \alpha^2}}$	$J_0(\alpha t)$
5.02	$\dfrac{1}{\sqrt{s^2 + \alpha^2}(\sqrt{s^2 + \alpha^2} + s)}$	$\dfrac{1}{\alpha} J_1(\alpha t)$
5.021	$\dfrac{1}{\sqrt{s^2 + \alpha^2}(\sqrt{s^2 + \alpha^2} + s)^n}$	$\dfrac{1}{\alpha^n} J_n(\alpha t)$ $\quad$ n is a non-negative integer.
5.031	$\dfrac{1}{s\sqrt{s^2 + \alpha^2}(\sqrt{s^2 + \alpha^2} + s)^n}$	$\dfrac{1}{\alpha^n} \displaystyle\int_0^t J_n(\alpha t) dt$ $\quad$ n is a non-negative integer.
5.04	$\dfrac{1}{\sqrt{s^2 + \alpha^2} + s}$	$\dfrac{1}{\alpha} \dfrac{J_1(\alpha t)}{t}$
5.041	$\dfrac{1}{(\sqrt{s^2 + \alpha^2} + s)^n}$	$\dfrac{n}{\alpha^n} \dfrac{J_n(\alpha t)}{t}$ $\quad$ n is a positive integer.
5.051	$\dfrac{1}{s(\sqrt{s^2 + \alpha^2} + s)^n}$	$\dfrac{n}{\alpha^n} \displaystyle\int_0^t \dfrac{J_n(\alpha t)}{t} dt$ $\quad$ n is a positive integer.
6.01	$\dfrac{1}{s} e^{-as}$	$u(t - a)$

6.02	$\frac{1}{s^2}e^{-as}$	$(t-a)u(t-a)$
6.03	$\left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}$	$tu(t-a)$
6.04	$\left(\frac{2}{s^3}+\frac{2a}{s^2}+\frac{a^2}{s}\right)e^{-as}$	$t^2u(t-a)$
6.05	$\frac{1}{s}(e^{-as}-e^{-bs})$ $a<b$	$u(t-a)-u(t-b)$
6.07	$\left(\frac{1-e^{-s}}{s}\right)^2$	$\begin{cases} t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & 2<t \end{cases}$
6.08	$\left(\frac{1-e^{-s}}{s}\right)^3$	$\begin{cases} 0.5t^2 & 0<t<1 \\ 0.75-(t-1.5)^2 & 1<t<2 \\ 0.5(t-3)^2 & 2<t<3 \\ 0 & 3<t \end{cases}$
6.09	$\frac{1}{s^2}(1-e^{-s})$	$\begin{cases} t & 0<t<1 \\ 1 & 1<t \end{cases}$
6.10	$\frac{1}{s^3}(1-e^{-s})^2$	$\begin{cases} 0.5t^2 & 0<t<1 \\ 1-0.5(t-2)^2 & 1<t<2 \\ 1 & 2<t \end{cases}$
6.21	$\frac{1}{s(1+e^{-s})}$	$\sum_{k=0}^{\infty}(-1)^k u(t-k)$

C. FUNCTION-TRANSFORM PAIRS* — *continued*

No.	$F(s)$	$f(t)$ $\quad 0 \leqq t$
6.22	$\dfrac{1}{s \sinh s}$	$2\sum_{k=0}^{\infty} u(t-2k-1)$
6.23	$\dfrac{1}{s \cosh s}$	$2\sum_{k=0}^{\infty} (-1)^k u(t-2k-1)$
6.24	$\dfrac{1}{s} \tanh s$	$u(t) + 2\sum_{k=1}^{\infty} (-1)^k u(t-2k)$ or $\sum_{k=0}^{\infty} (-1)^k u(t-2k) u(2k+2-t)$
6.25	$\dfrac{e^s - s - 1}{s^2(e^s - 1)}$	$t - \sum_{k=1}^{\infty} u(t-k)$ or $\sum_{k=0}^{\infty} (t-k) u(t-k) u(k+1-t)$

AUXILIARY PAIRS FOR TREATMENT OF JUMP FUNCTIONS

No.	$F(s)$	$f(x)$ $\quad 0 \leqq x$
8.01	$\dfrac{1-e^{-s}}{s} \triangleq P(s)$	$\left\{ \begin{array}{l} u(x) - u(x-1) \\ \text{or } u(x)u(1-x) \end{array} \right\} \triangleq p(x)$
8.02	$e^{-\gamma s} P(s)$	$p(x-\gamma)$
8.03	$\dfrac{e^s P(s)}{e^s - 1}$	1

JUMP-FUNCTION PAIRS

No.	$F(s)$	$f(x)$	$0 \leqq x$
8.11	$\dfrac{e^sP(s)}{(e^s-1)^2}$	$\int x$	
8.12	$\dfrac{e^sP(s)}{(e^s-1)^3}(e^s+1)$	$\int x^2$	
8.13	$\dfrac{e^sP(s)}{(e^s-1)^4}(e^{2s}+4e^s+1)$	$\int x^3$	
8.14	$\dfrac{e^sP(s)}{(e^s-1)^5}(e^{3s}+11e^{2s}+11e^s+1)$	$\int x^4$	
8.22	$\dfrac{e^sP(s)}{(e^s-1)^3}$	$\int \dfrac{x(x-1)}{2!} \triangleq \int \dfrac{x^{[2]}}{2!}$	
8.23	$\dfrac{e^sP(s)}{(e^s-1)^4}$	$\int \dfrac{x(x-1)(x-2)}{3!} \triangleq \int \dfrac{x^{[3]}}{3!}$	
8.24	$\dfrac{e^sP(s)}{(e^s-1)^{n+1}}$	$\int \dfrac{x(x-1)(x-2)\cdots(x-n+1)}{n!} \triangleq \int \dfrac{x^{[n]}}{n!}$	n is a non-negative integer.
8.30	$\dfrac{e^sP(s)}{e^s-c}$	$\int c^x$	c is a real number.
8.31	$\dfrac{e^sP(s)}{(e^s-c)^2}$	$\int xc^{x-1}$	c is a real number.
8.32	$\dfrac{e^sP(s)}{(e^s-c)^3}$	$\int \dfrac{x(x-1)c^{x-2}}{2!} \triangleq \int \dfrac{x^{[2]}c^{x-2}}{2!}$	c is a real number.

C. FUNCTION-TRANSFORM PAIRS* — *continued*

Jump Function Pairs — *continued*

No.	$F(s)$	$f(x)$	$0 \leqq x$
8.34	$\dfrac{e^s P(s)}{(e^s - c)^{n+1}}$	$\int \dfrac{x^{[n]} c^{x-n}}{n!}$	c is a real number, n is a non-negative integer.
8.40	$\dfrac{e^s P(s)}{(e^s - c)(e^s - d)}$	$\int \dfrac{c^x - d^x}{c - d}$	c and d are real numbers.
8.51	$\dfrac{e^s P(s)}{e^{2s} - 2(\cos\beta)e^s + 1}$	$\int \dfrac{\sin \beta x}{\sin \beta}$	
8.52	$\dfrac{(e^s - \cos\beta)e^s P(s)}{e^{2s} - 2(\cos\beta)e^s + 1}$	$\int \cos \beta x$	
8.61	$\dfrac{e^s P(s)}{(e^s - 1)^2(e^s - c)}$	$\int \dfrac{c^x - 1}{(c-1)^2} - \int \dfrac{x}{c-1}$	c is a real number.
8.74	$e^{as} \dfrac{e^s P(s)}{(e^s - 1)^{n+1}}$	$\int \dfrac{(x+a)^{[n]}}{n!}$	a and n are non-negative integers. $a \leqq n$
8.741	$e^{-as} \dfrac{e^s P(s)}{(e^s - 1)^{n+1}}$	$\int \dfrac{(x-a)^{[n]}}{n!} u(x-a)$	a and n are non-negative integers.
8.752	$e^{-as} \dfrac{e^s P(s)}{(e^s - c)^{n+1}}$	$\int \dfrac{(x-a)^{[n]} c^{x-a-n}}{n!} u(x-a)$	c is a real number, a and n are non-negative integers.

* α, β, γ, δ, and λ are real numbers.

APPENDIX B

COMPARISON OF THE LAPLACE AND THE FOURIER-INTEGRAL TRANSFORMATION METHODS

In Chapter 1, Art. 10, the usual Fourier-integral method was included in the brief comparison of methods for obtaining the complete solution of linear constant-coefficient integrodifferential equations. In Chapter 3 important formal relations between the Laplace and the Fourier-integral transformations were developed in more detail. The purpose of this appendix is to round out the comparison of these two integral transformations.

The principal point to be emphasized is that the usual Fourier integral in unilateral form is the form to which the type of Laplace integral used in the text reduces for $\sigma = 0$. The Laplace integral has been called a "generalized" or "complex" Fourier integral [HA 1, PA 1, TI 3]. However, as indicated in Appendix C, this integral was called the Laplace integral many years before the Fourier integral was generalized from imaginary to complex kernel exponent.

Of course, the unilateral form of the Fourier integral can be used in the same way as the Laplace to bring in initial conditions. Nevertheless, with a single recent exception [TI 3], it is not so used. This use is not considered in the most important engineering contribution on Fourier integrals [CA 2]. The current method for treating initial or boundary conditions by the Fourier integral (see for example [ST 4]) is that introduced by Cauchy in 1823 [CA 11].

As shown in Chapter 3, the Laplace integral is the natural and convenient generalization of an ordinary Fourier integral which makes possible the transformation of functions of growing exponential type that would otherwise be excluded. If a bilateral transformation is needed for treating such functions, then, as in [TI 3], it can be composed of two unilateral Laplace transformations with convergence factors directed toward $-\infty$ and ∞.

Those acquainted with the inverse Laplace and the Fourier-integral transformations will recognize that, in the language of the text, unless the path of integration is completely on the imaginary axis the inverse transformation is really of Laplace type. Usually then, the widely used contour integrals are Laplace integrals or such integrals with modified paths of integration.

To summarize, that form of transformation should be used which conveniently handles the class of functions concerned and simplifies the solution of problems, whether the transformation is Fourier, Laplace, Hankel, Watson, or even a more general type. For the class of functions and type of problems treated in the text, the unilateral Laplace transformation fits better than the unilateral or bilateral Fourier-integral transformation. Where the Laplace transformation reduces to the Fourier, this reduction is often of value, as in many network-synthesis problems.

Appendix C

HISTORICAL NOTES ON THE MATHEMATICAL THEORY

A. GENERAL REMARKS

The Laplace transformation and its applications are spread through so much of mathematics, mathematical physics, and mathematical engineering that a complete history of its evolution would require a separate volume. In general, such a history would cover functional transformations or operators and, in particular, such topics as the Dirichlet-series and Laplace-integral transformations, the Fourier-series and -integral transformations, and the Cauchy-Heaviside operational calculus.

These notes are intended to indicate where important material for such a history can be found insofar as it concerns those aspects of the subject which are touched upon in this text, and to help clear away some of the historical errors found in the existing literature, possibly at the risk of introducing others. Engineering literature in this general field is full of historical inaccuracies. Mathematical literature, even that which technically measures up to the standards of rigor of its day, is punctuated with historical shortcomings.

Another purpose of these notes is to supply newcomers in this field with historical perspective. The subject seems to give them two impressions, (1) that here is a " powerful tool," and (2) that it has been inadequately presented. The result often is a paper by another author who has not acquainted himself with a significant part of the literature, which is already enormous.

B. BROAD REFERENCES

General historical treatments have been given by Stephens [St 2] on operator theory up to 1900, Pincherle and Amaldi [Pi 5] on operator theory up to 1901, Pincherle [Pi 3] on functional operators and equations and the Laplace transformation up to 1902, Bateman [Ba 2] on integral equations up to 1910, and Doetsch [Do 5] on functional analysis up to 1927. The following should be consulted for their bibliographies: Doetsch [Do 15], Titchmarsh [Ti 3], McLachlan [Mc 2], and Widder [Wi 3].

C. THE LAPLACE TRANSFORMATION

In effect, Laplace [LA 2] introduced in 1779 the transformation which has received his name by showing the correspondence between the two functional domains which it relates (see Pincherle [PI 2] and Bateman [BA 2]). As Poole [Po 9, p. 137] has indicated, the Laplace transformation may be viewed as a limiting form of a transformation used by Euler. In this sense the transformation is related to work which preceded Laplace's. The definite integral form of the direct Laplace transformation was applied by Laplace [LA 3] to the solution of differential and difference equations in 1782. Laplace's book [LA 4], 1812, on the theory of probability illustrates the many uses to which he put the transformation.

From the Laplace double integral Cauchy [CA 15] first derived the Fourier double integral in exponential form. Many of the important formal properties of the Laplace transformation were discussed by Abel [AB 1]. His contact with this subject is reflected by the fact that numerous writers have referred to the transformation by his name alone or jointly with Laplace's.

Notable contributions subsequently were made to the theory and applications of the Laplace transformation by Poincaré [Po 1, 2], Pincherle [PI 2], Mellin [ME 1, 2, 3], and Lerch [LE 2].

Extensions, refinements, and applications were later contributed by Horn [Ho 2*a*, 3], Borel [Bo 6, 6*a*], Pincherle [PI 3*a* to 4*b*], Pisati [PI 10], Landau [LA 1*a*, 1*b*], Bromwich [BR 0], Mellin [ME 4], Bateman [BA 3, 4], and Kameda [KA 1].

These were followed by Hardy [HA 4 to 8], Hamburger [HA 3*a*], F. Bernstein [BE 4 to 8], Mellin [ME 5, 6], Doetsch [Do 1 to 10], Horn [Ho 4], Pollard [Po 8], von Stachó [ST 1], Haar [HA 1], Tamarkin [TA 1], Plancherel [PL 1, 2], Widder [WI 1], and Bochner [Bo 1].

During the past 10 years work not too distantly related to the material in this text was done by Hille and Tamarkin [HI 0, 0*a*, 1], Mächler [MA 1], Widder [WI 2, 2*a*, 3], Haviland [HA 9*a*], Paley and Wiener [PA 1], Doetsch [Do 15], Ignatovskij [IG 1], Titchmarsh [TI 3], Levi [LE 3], Churchill [CH 1 to 4], Obrechkoff [OB 1], Boas and Widder [Bo 0], and H. Pollard [Po 8*a*, 8*b*].

D. TYPES OF LAPLACE TRANSFORMATION

Besides the transformation with kernel of the form e^{-st}, Laplace [LA 2, p. 66; 3, p. 236; 4] used the alternative form with kernel of the form t^s. The latter form was preferred by Mellin, Hardy, and others. Hardy called it "the Mellin transformation," but owing to its close

connection with the base-e form and to its historical origin the name "Laplace-Mellin transformation" seems better. Of course, its relation to the base-e form corresponds to the relation of a power series to a Dirichlet series.

In the Laplace type, the singly infinite range of integration (unilateral form) may be used as in this text, or the doubly infinite range (bilateral form) may be used for other purposes. The functions and their transforms may be real or complex functions of real or complex variables. The type of integral may be Riemann, Lebesgue, Young-Stieltjes, or others. Also the s-multiplied or other forms of summable integrals may be used. Nothing will be said about the histories of the various forms not used in this text, although many of the recent mathematical developments have been concerned with these and other generalizations. The Lebesgue form of Laplace integral used here was used by Hardy [HA 5], 1918, if not earlier.

E. RELATIONS BETWEEN THE LAPLACE TRANSFORMATION AND THE CAUCHY-HEAVISIDE OPERATIONAL CALCULUS

Those acquainted with both techniques are aware of the close correspondence between the Cauchy-Heaviside operational calculus and the Laplace-transformation method. The use of the s-multiplied form of Laplace transformation (see for example [HA 3a]) provides complete formal isomorphism, while the type of transformation used in the text gives an isomorphism to within a factor of s in the complex domain, or essentially a differentiation (or integration) in the real domain.

The following remarks are intended to throw some light on the historical origin of the above-mentioned relation. Stephens [ST 2] and others have observed that the Cauchy-Heaviside calculus has numerous points in common with the earlier operational methods which originated with Leibnitz. About 1821, Cauchy [CA 11, 16] developed an operational calculus which, no doubt, was inspired partly by the earlier methods. Cauchy's operational calculus was based by him upon the Laplace and Fourier transformations, and is formally identical with parts of the operational calculus used more than 60 years later by Heaviside [HE 1, 2]. In the years which followed, Boole [Bo 4, 5], W. R. Hamilton, and others continued to develop operational methods. During this interval Cauchy [CA 12 to 15, 17] continued to extend his method and connect it in more ways with his work on functions of a complex variable. Except for Hargreave [HA 9] most of the writers on operational methods in this period forgot or ignored the relations between these methods and the Laplace and Fourier transformations. This remark applies in the main to the work of Heaviside, who mentioned so infrequently the

related work which preceded his that it can only be surmised how much he had learned from earlier writers, such as Boole, and how many known ideas he rediscovered. Heaviside certainly deserves credit for the extensive applications which he made of the Cauchy type of operational calculus to electrical problems. On this basis it is fitting to call this calculus and its extensions "the Cauchy-Heaviside operational calculus." Murnaghan [Mu 3] has used the double name for a relation in this calculus. Because Heaviside referred so rarely to the earlier literature on operators he has been credited by many writers with the invention of the operational calculus which he used. Also because he did not avail himself of the contemporary mathematical rigor he invited a host of later writers to supply rigorous foundations for the methods which he employed. It is now clear that Cauchy had not only supplied the original operational calculus of the type considered but had derived it by using the Laplace transformation. He had thereby supplied a basis for its rigorous treatment.

Among those who have attacked the problem of supplying rigorous foundations for all or parts of Heaviside's method are the following: Giorgi [Gi 2 to 5], Bromwich [Br 1], Wagner [Wa 4, 6], Pleijel and Liljeblad [Pl 2*a*], Carson [Ca 6, 8], Wiener [Wi 4], Lévy [Le 5], von Stachó [St 1], Murnaghan [Mu 3], Jeffreys [Je 1, 2], Korn [Ko 3, 4, 5], March [Ma 5], Plancherel [Pl 1, 2], Bush [Bu 1], Carslaw [Ca 4, 5*a*], Doetsch [Do 7, 10, 11], van der Pol [Po 4 to 7], Terradas [Te 1], Dalzell [Da 3], Schulz [Sc 4], von Koppenfels [Ko 2], Kryloff [Kr 1], Niessen [Po 5*b*, 6, 7], Lowry [Lo 6], Mächler [Ma 1], Vahlen [Va 1], Humbert [Hu 1, 2] Lowan [Lo 1], Angelini [An 1], Barnes [Es 1], Malti [Ma 3], Dahr [Da 1, 2], Poole [Po 9], Völlm [Vo 1], Blondel [Bl 2], Bourgin and Duffin [Bo 8, 9], Ekelöf [Ek 1], Pipes [Pi 6, 7, 9], Fujiwara [Fu 2], McLachlan [Mc 2], Carslaw and Jaeger [Ca 5*b*, 5*f*], Plessner [Pl 3], and Jaeger [Ja 2, 4].

As noted above, use of the Laplace transformation to supply a rigorous foundation for the C–H operational calculus was first made by Cauchy himself. The first contribution to appear after Heaviside's early work is apparently that of Giorgi [Gi 2 to 5]. Following this, the ordinary and *s*-multiplied forms of Laplace transformation — at times only the inverse or only the direct—were applied by Heaviside [He 2 vol. 3, pp. 235–237], Bromwich [Br 1], Wagner [Wa 4], Carson [Ca 6, 8], von Stachó [St 1], Jeffreys [Je 1], Plancherel [Pl 1, 2], Bush [Bu 1], van der Pol [Po 4], Doetsch [Do 7, 10, 11], Terradas [Te 1], von Koppenfels [Ko 2], Mächler [Ma 1], Lowan [Lo 1 to 5*a*], Barnes [Es 1] Bourgin and Duffin [Bo 8, 9], Pipes [Pi 6 to 9*a*], Droste [Dr 1 to 3], McLachlan [Mc 2], Carslaw and Jaeger [Ca 5*b* to 5*f*], and many others.

The closely related Fourier transformation [Appen B] was used by both Cauchy [Ca 11] and Heaviside [He 2, vol. 2, p. 32] in connection with their operational calculus. Later it was used to make formal work in this field rigorous, notably by Giorgi [Gi 2 to 5], Bromwich [Br 1], Fry [Fr. 3], Wiener [Wi 4], Haar [Ha 1], Campbell and Foster [Ca 2], Bush [Bu 1], von Koppenfels [Ko 2], Bochner [Bo 1], Paley and Wiener [Pa 1], Hille and Tamarkin [Hi 0, 1*a*], and Titchmarsh [Ti 3].

F. NOTES ON THE THEOREMS OF THE TEXT

1. Theorem 2, Inverse Laplace Transformation

This theorem on the integral inversion of the Laplace transformation came from a formal relation given by Riemann [Ri 1] in 1859. The history of this transformation was given by Mellin [Me 1], 1896; Bateman [Ba 2], 1910; and Hamburger [Ha 3*a*], 1920. Unfortunately, in Hamburger's account the fundamental work of Mellin is not mentioned. These three histories cite the work of Cauchy, 1851; Petzval, 1853; Riemann [Ri 1], 1859; Phragmén, 1891, 1892; Mellin [Me 1], 1896; Hadamard, 1908; Perron, 1908; and Landau, 1909. To this list may be added Kronecker, Dirichlet (see Pincherle [Pi 3*b*, 4*a*]), Mellin [Me 4], Hardy [Ha 5, 7, 8], Hamburger [Ha 3*a*], Pollard [Po 8], Izumi [Iz 1], and Hardy and Titchmarsh [Ha 8*a*].

The reference made by certain authors (Bromwich [Br 1], for example) to the derivation of the inversion formula given by Macdonald [Ma 0] in 1902 hardly seems warranted in view of the earlier formal derivation by Riemann [Ri 1] and to the proof by Mellin [Me 1], to mention only two of the papers which preceded Macdonald's.

Similarly, it is sometimes stated in a way which implies that the result was not known earlier that March [Ma 5], 1927, proved " Bromwich's contour integral " to be the solution of " Carson's integral equation," although the first is the integral inverse Laplace transformation and the second is the direct Laplace transformation.

Because of Mellin's work on the integral inverse Laplace transformation this transformation has been called by Hardy " the Mellin inversion formula." Others have used the names of both Cahen and Mellin.

New methods for inverting the Laplace transformation were developed by Stieltjes, Widder [Wi 2 to 3], Boas [Bo 0], Pollard [Po 8*a*, 8*b*], and others.

The linearity of the indicated inverse Laplace transformation comes from the work of Abel [Ab 1], who used a symbol to indicate the transformation.

2. Theorems 3 and 4, Unicity

These theorems came from generalization of a theorem of Lerch [Le 2]. The detailed history of this theorem was given by Landau [La 1*b*] and Bateman [Ba 2], where mention is made of the contributions of Liouville, 1837; Lerch, 1892, 1893, 1903 [Le 2]; Stieltjes, 1893; Phragmén [Ph 1], 1904; Landau, 1908; C. N. Moore, 1908; and W. H. Young, 1910. Later a note by Hardy [Ha 4] on this theorem referred to the work of Lebesgue, 1909, and Hobson, 1912.

3. Theorem 9, Complex Multiplication or Real Convolution

A real convolution integral probably appeared first in Euler's solution of a linear differential equation. In an algebraic form, the corresponding relation occurred in Cauchy's work on the multiplication of power series. Both convolution and the particular type of superposition using step functions, which is employed in the Cauchy-Heaviside calculus, appeared in a paper presented in 1826 by Poisson [Po 3]. The treatment in this paper leads to the supposition that these ideas were already known. In 1833 Duhamel [Du 1, 2] used both the superposition (addition) of step functions and convolution integrals. Duhamel's name is sometimes attached both to the superposition method and to convolution theorems, but this does not seem justified in either case. Murnaghan [Mu 3] and others associate this method of superposition with the names of Boltzmann [Bo 3] and Hopkinson [Ho 2], whose papers appeared in 1874 and 1877, respectively. Tchebycheff made implicit use of convolution in an 1867 paper [Tc 1] and explicit use in an 1887 paper [Tc 2]. Sonine [So 1], in 1884, writing on the inversion of a definite integral of Abel, used convolution. Attention should also be called to the work of Pareto [Pa 2] in 1892. Mellin [Me 2, 3], in 1896, used the relation between convolution and the Laplace transformation and proved convolution theorems for the cases of one and of several variables. He also used convolution in the solution of certain differential equations.

The real convolution theorem is usually named after Borel [Bo 6, 6*a*], 1899. Even if we restrict ourselves to naming a theorem after the one who first supplied a rigorous proof it would seem that Borel's name should not be attached to this theorem in view of the work of Mellin. Cailler [Ca 0], 1899, used convolution in the same connection as Sonine and also in combining the solutions of two Laplace differential equations.

More general forms of the real convolution theorem were later given by Cunningham [Cu 0], Bromwich [Br 0], Mellin [Me 4, 5], Horn [Ho 3],

Hardy [HA 6, 7, 8], F. Bernstein [BE 4, 5, 6], Doetsch [Do 1], von Koppenfels [Ko 2] and many later writers who used more general types of integrals than are considered in this text.

Generalizations of convolution in a somewhat different direction are due to Volterra. He calls the more general operation "composition." The history and uses of composition are discussed in Volterra's book [Vo 2].

4. Theorem 19, Real Multiplication or Complex Convolution

The earliest use of complex convolution is perhaps that made by Pincherle [PI 4] in 1908. This was made in connection with the unilateral Laplace transformation. Mellin [ME 4] stated the theorem without proof for the alternative form of the bilateral transformation in 1910. In 1922, Mellin [ME 5] proved his theorem under fairly general conditions.

The multiplication or convolution theorems in the Laplace-transformation theory should be contrasted with the corresponding theorems in the symmetrical L_2 Fourier-transformation theory. In the former each theorem has a distinct proof and history, while in the latter it would be difficult to think of one theorem without the other, and both follow simply from Parseval's theorem for the Fourier integral [WI 5, p. 70].

5. Impulse Function

Lewis [LE 7] gave a history of the impulse function. This type of function has been used for well over a hundred years but, ignoring this history, recent works on atomic physics often refer to it as "the Dirac δ function." (Compare Schönberg [Sc 3].)

G. SOLUTION OF DIFFERENTIAL, DIFFERENCE, AND INTEGRAL EQUATIONS BY THE LAPLACE TRANSFORMATION

Laplace [LA 3], 1782, was the first to use his integral in the solution of linear differential and difference equations. Bateman [BA 2] discussed Laplace's method and referred to extensions by Heine, Pincherle, Jordan, Pochhammer, and Schliesinger. Cauchy [CA 11, 12] and Abel [AB 1] made significant contributions to the solution of linear equations with constant coefficients. Papers in the more general field came from Poincaré [Po 1, 2], Pincherle [PI 1, 2, 3], Mellin [ME 2], Horn [Ho 2*a*, 3, 4], Cailler [CA 0], Cunningham [CU 0], and Bateman [BA 3, 4].

New life was given to the subject by a series of papers by F. Bernstein and Doetsch [BE 4 to 8, Do 1 to 7, 10, 11, 14, 15] who used the Laplace

transformation and its inverse in a straightforward way to solve linear integrodifferential equations of the convolution type, and linear partial differential equations arising in heat conduction and electric cable problems.

Other papers which also made use of the Laplace transformation for the solution of differential and integral equations are those of von Stachó [St 1], Plancherel [Pl 1, 2], van der Pol and Niessen [Po 4 to 7], Dalzell [Da 3], Koizumi [Ko 1], von Koppenfels [Ko 2], Mächler [Ma 1], Humbert [Hu 2], Lowan [Lo 1 to 5*a*], Bourgin and Duffin [Bo 8, 9], Pipes [Pi 6, 7, 9*a*], Droste [Dr 1 to 3], McLachlan [Mc 2], Churchill [Ch 1 to 4], Wagner [Wa 6, 7], Söhngen [So 2], and Carslaw and Jaeger [Ca 5*a* to 5*f*, Ja 2 to 5].

The solution of difference, differential, and integral equations by the Fourier transformation [Appen B] should be mentioned because of its close relation to the Laplace-transformation method. Cauchy [Ca 11], Giorgi [Gi 2 to 5], Fry [Fr 3], Wiener [Wi 4], Campbell and Foster [Ca 2], von Koppenfels [Ko 2], Bochner [Bo 1, Chap. 5], and Titchmarsh [Ti 3] are among those who have used the Fourier transformation for this purpose.

H. INVERSE TRANSFORMATION OF A RATIONAL ALGEBRAIC FUNCTION

As illustrated in the text, this topic arises in the solution of ordinary differential or i-d equations. The solution of the problem goes back to Cauchy [Ca 11, p. 583; Ca 16, pp. 217, 218; Ca 14, p. 228]. Later Heaviside [He 1, 2] also gave several derivations of the partial-fraction expansion method for finding the inverse transform [Va 2]. The theorem is widely known as "the Heaviside partial-fraction expansion theorem." However, with Murnaghan [Mu 3], we prefer to call it "the Cauchy-Heaviside partial-fraction expansion theorem." Together with its generalizations this theorem has been discussed by a large number of writers. Some of the papers are by Bromwich [Br 1], Wagner [Wa 4], Carson [Ca 8], Cohen [Co 1], Gotō [Go 1], Plancherel [Pl 1, 2], Berg [Be 3], Doetsch [Do 8], and Bourgin and Duffin [Bo 8].

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